JEE Advanced (2022)

PAPER-II

Mathematics

SECTION 1 (Maximum Marks: 24)

- This section contains **EIGHT (08)** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER ranging from 0 to 9, BOTH INCLUSIVE**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct integer is entered;

Zero Marks : 0 If the question is unanswered;

Negative Marks : –1 In all other cases.

Q. 1. Let α and β be real numbers such that $-\frac{\pi}{4} < \beta < 0 < \alpha < \frac{\pi}{4}$. If $\sin(\alpha + \beta) = \frac{1}{3}$ and $\cos(\alpha - \beta) = \frac{2}{3}$, then the greatest integer less than or equal to

$$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\cos\alpha}{\sin\beta} + \frac{\sin\beta}{\cos\alpha}\right)^2$$

is ________.

Q. 2. If $y(x)$ is the solution of the differential equation $xdy - (y^2 - 4y)dx = 0$ for $x > 0$, $y(1) = 2$, and the slope of the curve $y = y(x)$ is never zero, then the value of $10\, y(\sqrt{2})$ is ________.

Q. 3. The greatest integer less than or equal to $\int_1^2 \log_2(x^3+1)dx + \int_1^{\log_2 9}(2^x - 1)^{\frac{1}{3}}dx$ is ________.

Q. 4. The product of all positive real values of x satisfying the equation

$$x^{(16(\log_5 x)^3 - 68 \log_5 x)} = 5^{-16}$$

is ________.

Q. 5. If $\beta = \lim_{x \to 0} \dfrac{e^{x^3} - (1 - x^3)^{\frac{1}{3}} + \left((1 - x^2)^{\frac{1}{2}} - 1\right)\sin x}{x \sin^2 x}$,

then the value of 6β is ________.

Q. 6. Let β be a real number. Consider the matrix $A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$. If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is ________.

Q. 7. Consider the hyperbola

$$\frac{x^2}{100} - \frac{y^2}{64} = 1$$

with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta = S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is ________.

Q. 8. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + \frac{5}{12}$ and

$$g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right), & |x| \le \frac{3}{4}, \\ 0, & |x| > \frac{3}{4}. \end{cases}$$

If α is the area of the region

$\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \le \frac{3}{4}, 0 \le y \le \min\{f(x), g(x)\}\}$,

then the value of 9α is ________.

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4 **ONLY** if (all) the correct option(s) is (are) chosen;
Partial Marks	:	+3 If all the four options are correct but **ONLY** three options are chosen;
Partial Marks	:	+2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks	:	+ If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks	:	0 If unanswered;
Negative Marks	:	–2 In all other cases.

Q. 9. Let $PQRS$ be a quadrilateral in a plane, where $QR = 1$, $\angle PQR = \angle QRS = 70°$, $\angle PQS = 15°$ and $\angle PRS = 40°$. If $\angle RPS = \theta$, $PQ = \alpha$ and $PS = \beta$, then the interval(s) that contain(s) the value of $4\alpha\beta \sin \theta°$ is/are

(A) $(0, \sqrt{2})$ **(B)** $(1, 2)$

(C) $(\sqrt{2}, 3)$ **(D)** $(2\sqrt{2}, 3\sqrt{2})$

Q. 10. Let

$$\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right).$$

Let $g : [0, 1] \to \mathbb{R}$ be the function defined by

$$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}.$$

Then, which of the following statements is/are TRUE ?

(A) The minimum value of $g(x)$ is $2^{\frac{7}{6}}$

(B) The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$

(C) The function $g(x)$ attains its maximum at more than one point

(D) The function $g(x)$ attains its minimum at more than one point

Q. 11. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of

$$(\bar{z})^2 + \frac{1}{z^2}$$

are integers, then which of the following is/are possible value(s) of $|z|$?

(A) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ **(B)** $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(C) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ **(D)** $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

Q. 12. Let G be a circle of radius $R > 0$. Let $G_1, G_2, ..., G_n$ be n circles of equal radius $r > 0$. Suppose each of the n circles $G_1, G_2, ..., G_n$ touches the circle G externall. Also, for $i = 1, 2, ..., n - 1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE ?

(A) If $n = 4$, then $(\sqrt{2} - 1)r < R$

(B) If $n = 5$, then $r < R$

(C) If $n = 8$, then $(\sqrt{2} - 1)r < R$

(D) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$

Q. 13. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along the three positive coordinate axes. Let

$$\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$$
$$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}, \qquad b_2, b_3 \in \mathbb{R},$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

be three vectors such that $b_2b_3 > 0$, $\vec{a} \cdot \vec{b} = 0$ and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}.$$

Then, which of the following is/are TRUE ?

(A) $\vec{a} \cdot \vec{c} = 0$ **(B)** $\vec{b} \cdot \vec{c} = 0$

(C) $|\vec{b}| > \sqrt{10}$ **(D)** $|\vec{c}| \le \sqrt{11}$

Q 14. For $x \in \mathbb{R}$, let the function $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right), \quad y(0) = 0$$

Then, which of the following statements is/are TRUE ?

(A) $y(x)$ is an increasing function

(B) $y(x)$ is a decreasing function

(C) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points

(D) $y(x)$ is a periodic function

SECTION 3 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each set has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (*i.e.*, the question is unanswered);

Negative Marks : –1 In all other cases.

Q. 15. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen ?

(A) 21816 **(B)** 85536

(C) 12096 **(D)** 156816

Q. 16. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022} ?

(A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ **(B)** $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$ **(D)** $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

Q. 17. Suppose that

Box-I contains 8 red, 3 blue and 5 green balls,

Box-II contains 24 red, 9 blue and 15 green balls,

Box-III contains 1 blue, 12 green and 3 yellow balls,

Box-IV contains 10 green, 16 orange and 6 white balls.

A ball is chosen randomly from Box-I; call this ball b. If b is red then a ball is chosen randomly from Box-II, if b is blue then a ball is chosen randomly from Box-III, and if b is green then a ball is chosen randomly from Box-IV. The conditional probability of the event 'at least one of the chosen balls is white' given that the event 'at least one of the chosen balls is green' has happened, is equal to

(A) $\frac{15}{256}$ **(B)** $\frac{3}{16}$

(C) $\frac{5}{52}$ **(D)** $\frac{1}{8}$

Q. 18. For positive integer n, define

$$f(n) = n + \frac{16+5n-3n^2}{4n+3n^2} + \frac{32+n-3n^2}{8n+3n^2} + \frac{48-3n-3n^2}{12n+3n^2} + \ldots + \frac{25n-7n^2}{7n^2}$$

Then, the value of $\lim_{n\to\infty} f(n)$ is equal to

(A) $3+\frac{4}{3}\log_e 7$ **(B)** $4-\frac{3}{4}\log_e\left(\frac{7}{3}\right)$

(C) $4-\frac{4}{3}\log_e\left(\frac{7}{3}\right)$ **(D)** $3+\frac{3}{4}\log_e 7$

❑❑❑

Q. no.	Answer	Topic name	Chapter name
1	[1]	Properties of Trigonometric Identities	Trigonometric Functions
2	[8]	Solution of Differential Equations	Differential Equations
3	[5]	Properties of Definite Integral	Integration
4	[1]	Properties of log	Logarithm
5	[5]	Properties of limits	limit and Derivatives
6	[3]	Matrics and Determinants	Matrics and Determinants
7	[7]	Hyperbola	Two Dimensional (2D)
8	[6]	Area bounded the region	Application of Integral
9	A&B	Relation between sides and angles	Trigonometric Ratio
10	A,B&C	Maxima and Minima	Differentiation
11	A	Properties of modules	Complex Number
12	C&D	Circle	Two Dimensional (2D)
13	B,C&D	DOT and Cross products of vectors	Vector Algebra
14	C	Solution of differential equations	Differential Equations
15	A	Probability	Probability
16	A	Matrics and Determinants	Matrics and Determinants
17	C	Probability	Probability
18	B	properties of log	Logarithm

Answers

1. Correct answer is [1]

Explanation : Given $\alpha \in \left(0, \frac{\pi}{4}\right), \beta = \left(-\frac{\pi}{4}, 0\right)$

$$\Rightarrow \alpha + \beta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$\sin(\alpha+\beta) = \frac{1}{3}, \cos(\alpha-\beta) = \frac{2}{3}$$

$$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\alpha}{\sin\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\sin\beta}{\cos\alpha}\right)^2$$

$$\Rightarrow \left(\frac{\cos(\alpha-\beta)}{\cos\beta\sin\beta} + \frac{\cos(\beta-\alpha)}{\sin\alpha\cos\alpha}\right)^2$$

$$\Rightarrow 4\cos^2(\alpha-\beta)\left(\frac{1}{\sin 2\beta} + \frac{1}{\sin 2\alpha}\right)^2$$

$$\Rightarrow 4\cos^2(\alpha-\beta)\left(\frac{2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha \sin 2\beta}\right) \quad \ldots(1)$$

$$\Rightarrow \frac{16\cos^4(\alpha-\beta)\sin^2(\alpha+\beta)\times 4}{\{\cos 2(\alpha-\beta) - \cos 2(\alpha+\beta)\}^2}$$

$$\Rightarrow \frac{64\cos^2(\alpha-\beta)\sin^2(\alpha+\beta)}{\left\{2\cos^2(\alpha-\beta)-1-1+2\sin^2(\alpha+\beta)\right\}^2}$$

$$\Rightarrow 64 \times \frac{16}{81} \times \frac{1}{9} \times \frac{1}{\left(2\times\frac{4}{9}-1-1+\frac{2}{9}\right)^2}$$

$$\Rightarrow \frac{64\times16\times81}{81\times9\times64} = \frac{16}{9}$$

$$\left[\frac{16}{9}\right] = 1$$

2. Correct answer is [8]

Explanation : Given $xdy - (y^2 - 4y)dx = 0 \; x > 0$

$\Rightarrow xdy = (y^2 - 4y)\,dx$

$$\Rightarrow \frac{dy}{y^2 - 4y} = \frac{1}{x}dx$$

$$\Rightarrow \int \frac{dy}{y^2-4y} = \int \frac{1}{x}dx$$

(using variable separable method)

$$\Rightarrow \frac{1}{4}\int\left(\frac{1}{y-4} - \frac{1}{y}\right)dy = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{4}\left(\log\frac{y-4}{y}\right) = \log x + c$$

As, $y(1) = 2, c = 0$

So, $\frac{y-4}{y} = \pm x^4$

or $y = \frac{4}{1 \pm x^4}$

Consider $y = \frac{4}{1+x^4}$

So $10y\left(\sqrt{2}\right) = 8$

3. Correct answer is [5]

Explanation :

Let $f(x) = \log_2(x^3 + 1) = y$

then $x^3 + 1 = 2^y$

$x = (2^y - 1)^{1/3}$

$x = f^{-1}(y) = (2^y - 1)^{1/3}$

Put $x = y$

$f^{-1}(x) = (2^x - 1)^{1/3}$

Since $\int_1^2 \log_2(x^3+1)dx + \int_1^{\log_2 9}(2^x - 1)^{1/3}dx$

$\int_1^2 f(x)dx + \int_1^{\log_2 9} f^{-1}(x)dx$

$= 2\log_2 9 - 1$

$= 8 < 9 < 2^{7/2}$

$\Rightarrow 3 < \log_2 9 < \frac{7}{2}$

$= 5 < 2\log_2 9 - 1 < 6$

$5 = [2\log_2 9 - 1]$

4. Correct answer is [1]

Explanation : $x^{\left(16(\log_5 x)^3 - 68\log_5 x\right)} = 5^{-16}$

taking log to the base 5 on both sides

$\left[16(\log_5 x)^3 - 68(\log_5 x\right]\left(\log_5 x\right) = -16$

Let $\log_5 x = t$

or $4t^4 - 68t^2 + 16 = 0$

$\Rightarrow 4t^4 - 16t^2 - t^2 + 4 = 0$

$\Rightarrow (4t^2 - 1)(t^2 - 4) = 0$

$\Rightarrow t = \pm\frac{1}{2}, \pm 2$

So $\log_5 x = \pm\frac{1}{2}$ or ± 2

$\Rightarrow x = 5^{\frac{1}{2}}, 5^{-\frac{1}{2}}, 5^2, 5^{-2}$

$\therefore x_1 x_2 x_3 x_4 = 1$

5. Correct answer is [5]

Explanation : Given

$$\beta = \lim_{x\to 0}\frac{e^{x^3}-(1-x^3)^{1/3}+\left((1-x^2)^{1/2}-1\right)\sin x}{x\sin^2 x}$$

$$\beta = \lim_{x\to 0}\frac{e^{x^3}-(1-x^3)^{1/3}}{\frac{x\sin^2 x}{x^2}\cdot x^2}+\frac{\left\{(1-x^2)^{1/2}-1\right\}\sin x}{\frac{x.\sin^2 x}{x^2}\,x^2}$$

Use expansion

$$\beta = \lim_{x\to 0}\frac{(1+x^3)-\left(1-\frac{x^3}{3}\right)}{x^3}+\lim_{x\to 0}\frac{\left\{\left(1-\frac{x^2}{2}\right)-1\right\}}{x^2}\cdot\frac{\sin x}{x}$$

$$\beta = \lim_{x\to 0}\frac{4x^3}{3x^3}+\lim_{x\to 0}\frac{-x^2}{2x^2}$$

$$\beta = \frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

$$6\beta = 5$$

6. Correct answer is [3]

Explanation : Given $A = \begin{bmatrix}\beta & 0 & 1\\ 2 & 1 & -2\\ 3 & 1 & -1\end{bmatrix}$

$$|A| = -1$$

$$\Rightarrow \quad |A^7-(\beta-1)A^6-\beta A^5| = 0$$

$$\Rightarrow \quad |A|^5\,|A^2-(\beta-1)A-\beta I| = 0$$

$$\Rightarrow \quad |A|^5\,|A^2-\beta A)+A-\beta I| = 0$$

$$\Rightarrow \quad |A|^5\,|A(A-\beta I)+I(A-\beta I)| = 0$$

$$\Rightarrow \quad |A|^5|(A+I)(A-\beta I)| = 0$$

$$A+I = \begin{bmatrix}\beta & 0 & 1\\ 2 & 1 & -2\\ 3 & 1 & -1\end{bmatrix}$$

$$\Rightarrow \quad |A+I| = -4$$

Here $|A| \neq 0$ then $|A+I| \neq 0$

$$A-\beta I = \begin{bmatrix}0 & 0 & 1\\ 2 & 1-\beta & -2\\ 3 & 1 & -2-\beta\end{bmatrix}$$

$$\Rightarrow \quad |A-\beta I| = 2-3(1-\beta) = 3\beta-1 = 0$$

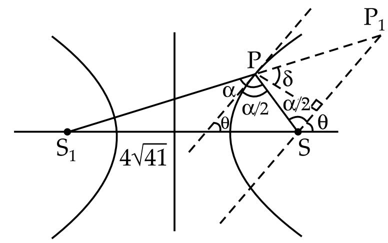
$$\Rightarrow \quad \beta = \frac{1}{3}$$

$$\therefore \quad 9\beta = 3$$

7. Correct answer is [7]

Explanation : $S_1P - SP = 20$

$$\beta - \frac{\delta}{\sin\frac{\alpha}{2}} = 20$$



$$\Rightarrow \beta^2 - \frac{\delta}{\sin^2\frac{\alpha}{2}} - \frac{2\beta\delta}{\sin\frac{\alpha}{2}} = 400$$

(Squaring both sides)

$$\Rightarrow \quad \beta^2 - \frac{\delta^2}{\sin^2\frac{\alpha}{2}} - 400 = \frac{2\beta\delta}{\sin\frac{\alpha}{2}}$$

$$\frac{1}{SP} = \frac{\sin\alpha/2}{\delta}$$

$$\cos\alpha = \frac{SP^2+\beta^2-256}{2\beta\frac{\delta}{\sin\frac{\alpha}{2}}}$$

$$\cos\alpha = \frac{\frac{2\beta\delta}{\sin\frac{\alpha}{2}}-256}{\frac{2\beta\delta}{\sin\frac{\alpha}{2}}}$$

But $\cos\alpha = \dfrac{\lambda-128}{\lambda}$

$$\Rightarrow \quad \frac{\beta\delta}{\sin\frac{\alpha}{2}}.2\sin^2\frac{\alpha}{2} = 128$$

(Put value of cos α}

$$\Rightarrow \quad \frac{\beta\delta}{\sin\frac{\alpha}{2}}.2\sin^2\frac{\alpha}{2} = 128$$

$$\Rightarrow \quad \frac{\beta\delta}{9}.\sin\frac{\alpha}{2} = \frac{64}{9}$$

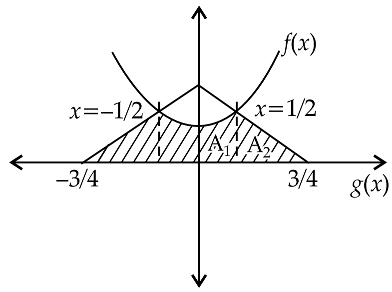
$$\Rightarrow \quad \frac{\beta\delta}{9}\sin\frac{\alpha}{2} = 7$$

8. Correct answer is [6]

Explanation : Given $f(x) = g(x)$

$$x^2+\frac{5}{12} = 2-\frac{8x}{3}$$

$\Rightarrow \quad x^2 + \frac{8x}{3} + \frac{5}{12} - 2 = 0$



$\Rightarrow \quad 12x^2 - 32x - 9 = 0$

$\Rightarrow \quad 12x^2 + 38x - 6x - 19 = 0$

$\Rightarrow \quad 2x(6x + 19) - 1(6x + 19) = 0$

$\Rightarrow \quad (6x + 19)(2x - 1) = 0$

$$x = \frac{1}{2}, \frac{-19}{6}$$

Rquired area, $\alpha = 2.(A_1 + A_2)$

$$\alpha = 2\left[\int_0^{1/2}\left(x^2 + \frac{5}{12}\right)dx + \frac{1}{2}\left(\frac{3}{4} - \frac{1}{2}\right) \times \frac{2}{3}\right]$$

$$\alpha = 2\left[\left|\left(\frac{x^3}{3} + \frac{5x}{12}\right)\right|_0^{1/2} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{3}\right]$$

$$\alpha = 2\left(\frac{1}{4} + \frac{1}{12}\right)$$

$$\alpha = \frac{2}{3}$$

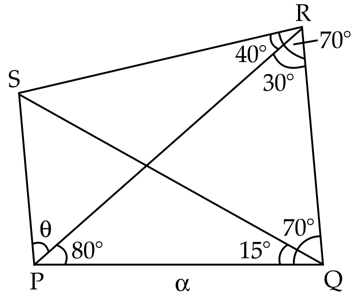
$9\alpha = 6$

9. Options (A) & (B) are correct.

Explanation : $\angle PRQ = 70° - 40° = 30°$

$\angle RQS = 70° - 15° = 55°$

$\angle QSR = 180° - 55° - 70°R = 55°$



$\therefore \quad QR = RS = 1$

$\angle QPR = 180° - 70° - 30° = 80°$

Apply sine- rule in ΔPRQ

$$\frac{\alpha}{\sin 30°} = \frac{1}{\sin 80°}$$

$$\Rightarrow \quad \alpha = \frac{1}{\alpha \sin 80°} \quad ...(1)$$

Apply sine rule in ΔPRS

$$\frac{\beta}{\sin 40°} = \frac{1}{\sin \theta}$$

$$\Rightarrow \quad \beta \sin \theta = \sin 40° \quad ...(2)$$

$$\Rightarrow \quad 4\alpha\beta \sin \theta = \frac{4 \sin 40°}{2 \sin 80°}$$

[using eq. (1) and (2)]

$$= \frac{4 \sin 40°}{2(2 \sin 40° \cos 40)}$$

$$= \sec 40° = \frac{1}{\cos 40°}$$

Now, $\quad \sec 30° < \sec 40° < \sec 45°$

$$\Rightarrow \quad \frac{2}{\sqrt{3}} < \sec 40° < \sqrt{2}$$

Using $\quad \cos 0° = 1,$

$$\cos 30° = \frac{\sqrt{3}}{2},$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$\cos 60° = \frac{1}{2}$$

10. Options (A), (B) & (C) are correct.

Explanation : Given $\alpha = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k}$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \frac{1/4}{1 - \frac{1}{4}} = \frac{1}{3}$$

Hence, $\quad g(x) = 2^{x/3} + 2^{(1-x)/3}$

Now, $\quad g'(x) = \frac{\log 2}{3}\left(\frac{(2^{2x/3} - 2^{1/3}}{2^{x/3}}\right)$

$$g'(x) = 0 \text{ at } x = \frac{1}{2}$$

And, derivative change sign from negative to positive at $x = \frac{1}{2}$, hence $x = \frac{1}{2}$ is point of local minimum as well as absolute minimum of g(x) for $x \in [0, 1]$

Hence, minimum value of

$$g(x) = g\left(\frac{1}{2}\right) = 2^{7/6}$$

Maximum value of $g(x)$ is either equal to $g(0)$ or $g(1)$

So $\quad g(0) = 1 + 2^{1/3}$

$\quad g(1) = 2^{1/3} + 1$

11. Option (A) is correct.

Explanation : Let $(\bar{z})^2 + \frac{1}{z^2} = m + ni \quad m, n \in z$

$$\Rightarrow \quad (\bar{z})^2 + \frac{\bar{z}^2}{|z|^4} = m + ni$$

$$\Rightarrow \quad (x^2 - y^2)\left(1 + \frac{1}{|z|^4}\right) = m \quad ...(1)$$

$$\text{and} \quad -2xy\left(1 + \frac{1}{|z|^4}\right) = h \quad ...(2)$$

Square Eq. (1) and (2)

$$\left(1 + \frac{1}{|z|^4}\right)^2 \left[(x^2 + y^2)^2\right] = m^2 + n^2$$

$$\Rightarrow \quad \left(1 + \frac{1}{|z|^4}\right)^2 (|z|)^4 = m^2 + n^2$$

$$\Rightarrow \quad |z|^4 + \frac{1}{|z|^4} + 2 = m^2 + n^2$$

For option (A)

$$|z|^4 = \frac{43 + 3\sqrt{205}}{2}$$

$$\Rightarrow \quad m^2 + n^2 = 45$$

$$\Rightarrow \quad m = \pm 6 \text{ and } n = \pm 3$$

For option (B)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{33}}{4} + \frac{7 - \sqrt{33}}{4} + 2$$

$$= \frac{7}{2} + 2 = \frac{11}{2}$$

For option (C)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{9 + \sqrt{65}}{4} + \frac{9 - \sqrt{65}}{4} + 2$$

$$= \frac{18}{4} + 2 = \frac{13}{2}$$

For option (D)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6} + 2$$

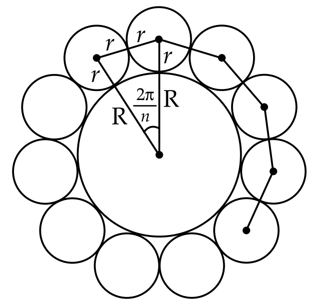
$$= \frac{14}{6} + 2$$

$$= \frac{13}{2}$$

12. Options (C) & (D) are correct.

Explanation : Since $2(R + r)\ \sin\frac{\pi}{n} = 2r$

$$\frac{R + r}{r} = \text{cosec}\ \frac{\pi}{n}$$



For option (A) $n = 4, R = \ R = r\,(\sqrt{2} - 1)$ False

For option (B) $n = 5,\ R = r\left(\text{cosec}\ \frac{\pi}{5} - 1\right)$

$$\Rightarrow \quad R < r \ \ R = r\left(\text{cosec}\ \frac{\pi}{5} - 1\right)$$

$\Rightarrow$

$R < r$ False

For option (C) $n = 8,\ R = r\left(\text{cosec}\ \frac{\pi}{8} - 1\right)$

$$\Rightarrow \quad R > r \ \ R > r\left(\text{cosec}\ \frac{\pi}{4} - 1\right)$$

$\Rightarrow \quad R > r\left(\sqrt{2} - 1\right)$ True

For option (D) $n = 12,\ R = r\left(\text{cosec}\ \frac{\pi}{12} - 1\right)$

$$\Rightarrow \quad R = \left[\sqrt{2}\,(\sqrt{3} + 1) - 1\right]r$$

$R < \sqrt{2}\,(\sqrt{3} + 1)r$ True

13. Options (B), (C) & (D) are correct.

Explanation : Given $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$

$$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\text{Given} \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{bmatrix}$$

Multiply and compare

$$b_2c_3 - b_3c_2 = c_1 - 3 \quad ...(1)$$

$$c_3 - b_3c_1 = 1 - c_2 \quad ...(2)$$

$$c_2 - b_2c_1 = 1 + c_3 \quad ...(3)$$

eq (1) $\times\ i$ – (2) $\times\ \hat{j}$ + (3) $\times\ \hat{k}$

$$\Rightarrow i\,(b_2c_3 - b_3c_2) - \hat{j}\,(c_3 - b_3c_1) + \hat{k}\,(c_2 - b_2c_1)$$

$$= c_1\hat{i} + c_2\hat{j} + c_2\hat{k} - 3\hat{i} - \hat{j} + \hat{k}$$

$\Rightarrow \quad \vec{d} \times \vec{c} = \vec{c} - \vec{a}$

(Taking dot product with $\vec{b}$)

$\Rightarrow \quad (\vec{c} \times \vec{b}).\vec{b} = (\vec{a} - \vec{c}).b$

$\vec{b} \cdot \vec{c} = 0$

So $\quad \vec{b} \perp \vec{c}$

Taking dot product with $\vec{c}$

$(\vec{c} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} - |\vec{c}|^2 = 0$

$\Rightarrow \quad 0 = \vec{a} \cdot \vec{c} - |\vec{c}|^2$

$\Rightarrow \quad |\bar{c}|^2 = \vec{a} \cdot \vec{c}$

$\Rightarrow \quad \vec{a} \cdot \vec{c} \neq 0$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Squaring both sides

$$|\vec{b}|^2 |\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c} \cdot \vec{a}$$

$$= |\vec{c}|^2 + 11 - 2|\vec{c}|^2$$

$$= 11 - |\vec{c}|^2$$

$$|\vec{b}|^2 |\vec{c}|^2 + |\bar{c}|^2 = 11$$

$$|\vec{c}|(|\vec{b}|^2 + 1) = 11$$

$$|\vec{c}|^2 = \frac{11}{1 + |\vec{b}|^2}$$

$\therefore \quad |\vec{c}| \leq \sqrt{11}$

Give $\quad \bar{a}.\bar{b} = 0$

So, $\quad b_2 - b_3 = -3$

Squaring both sides

$b_2^2 + b_3^2 - 2b_2b_3 = 9 \qquad \therefore b_2b_3 > 0$

$\Rightarrow \quad b_2^2 + b_3^2 = 9 + 2b_2b_3$

So $\quad b_2^2 + b_3^2 > 9$

$$|\vec{b}| = \sqrt{1 + b_2^2 + b_3^2}$$

$$|\vec{b}| > \sqrt{10}$$

14. Option (C) is correct.

Explanation : Given $\dfrac{dy}{dx} + 12y = \cos\left(\dfrac{\pi}{12}x\right)$

$$\text{I.F.} = e^{\int 12\,dx} = e^{12x}$$

Solution of differencial equation is

$$y \times \text{I.F.} = \int e^{12x}.\cos\left(\frac{\pi}{12}x\right)dx$$

$$ye^{12x} = \frac{e^{12x}}{12^2 + \left(\frac{\pi}{12}\right)^2}\left(12\cos\frac{\pi}{12}x + \frac{\pi}{12}\sin\frac{\pi}{12}x\right) + c$$

$$\Rightarrow y = \frac{12}{(12)^4 + \pi^2}\left\{(12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right)\right\} + \frac{c}{e^{12x}}$$

Given $\quad y(0) = 0$

$$\Rightarrow \quad 0 = \frac{12.(12^2)}{12^4 + \pi^2} + c$$

$$\Rightarrow \quad c = \frac{-12^3}{12^4 + \pi^2}$$

$$\therefore y = \frac{12}{12^4 + \pi^2}\left[12^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right) - 12^2.e^{-12x}\right]$$

Now, $\dfrac{dy}{dx} = \dfrac{12}{12^4 + \pi^2}$

$$\left[\underbrace{-12\pi\sin\left(\frac{\pi x}{12}\right) + \frac{\pi^2}{12}\cos\left(\frac{\pi x}{12}\right)}_{\text{min value}} + 12^3 e^{-12x}\right]$$

$$\left[-\sqrt{144\pi^2 + \frac{\pi^4}{144}} = -12\pi\sqrt{1 + \frac{\pi^2}{12^4}}\right]$$

$\Rightarrow \dfrac{dy}{dx} > 0 \;\; \forall\; x \leq 0$ and may be negative/positive for $\quad x > 0$

So, $f(x)$ is neither increase nor decrease

For some $\beta \in R$, $y = \beta$ intersects $y = f(x)$ at infinitely many points.

15. Option (A) is correct.

Explanation : Given

B-1	B-2	B-3	B-4
3R	3R	3R	3R
2B	2B	2B	2B

B → Bag

Case I: When exactly one box provides four ball (3R, 1B or 2R, 2B)

Number of ways in this case $= {}^5C_4({}^3C_1 \times {}^2C_1)^3 \times 4$

Case II: When exactly two boxes provide three balls (2R, 1B or 1R, 2B)

Number of ways in this case

$$= ({}^5C_3 - 1)^2\,({}^3C_1 \times {}^2C_1)^2 \times 6$$

Required number of ways = 21816

16. Option (A) is correct.

Explanation : Given $M = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$

$$M = \begin{bmatrix} \frac{3}{2}+1 & \frac{3}{2} \\ \frac{-3}{2} & \frac{-3}{2}+1 \end{bmatrix}$$

$$M = I + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since, $M^{2022} = \left(I + \frac{3}{2}A\right)^{2022}$

$$= I + 3033\,A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3033 & 3033 \\ -3033 & -3033 \end{bmatrix}$$

$$= \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$$

17. Option (C) is correct.

Explanation : Given

Box I	8(R)	3(B)	5(G)
Box II	24(R)	9(B)	15(G)
Box III	1(B)	12(G)	3(Y)
Box IV	10(G)	16(0)	6(w)

A (one of the chosen ball is white)

B (at least one of the chosen ball is green)

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{5}{16} \times \frac{6}{32}}{\frac{5}{16} \times 1 + \frac{8}{16} \times \frac{15}{48} + \frac{3}{16} \times \frac{12}{16}}$$

$$= \frac{15}{156} = \frac{5}{52}$$

18. Option (B) is correct.

Explanation : Given

$$f(n) = n + \sum_{r=1}^{n} \frac{16r + (9-4r)n - 3n^2}{4rn + 3n^2}$$

$$f(n) = n + \sum_{r=1}^{n} \frac{(16r + 9n) - (4rn + 3n^2)}{4rn + 3n^2}$$

$$f(x) = n + \left(\sum_{r=1}^{n} \frac{16r + 9n}{4rn + 3n^2}\right) - n$$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{16r + 9n}{4rn + 3n^2}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\left\{16\left(\frac{r}{n}\right) + 9\right\}\frac{1}{n}}{4\left(\frac{r}{n}\right) + 3}$$

$$= \int_0^1 \frac{16x + 9}{4x + 3}dx = \int_0^1 4dx - \int_0^1 \frac{3dx}{4x + 3}$$

$$= 4 - \frac{3}{4}\left[\left(\log_e (4x + 3)\right]_0^1\right.$$

$$= 4 - \frac{3}{4}\log_e \frac{7}{3}$$

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