JEE (Main) MATHEMATICS SOLVED PAPER

2023
24th Jan. Shift 1

General Instructions :

1. *In mathematics Section, there are 30 Questions (Q. no. 1 to 30).*
2. *In mathematics, Section A consists of 20 single choice questions & Section B consists of 10 numerical value type questions. In Section B, candidates have to attempt any five questions out of 10.*
3. *There will be only one correct choice in the given four choices in Section A. For each question for Section A, 4 marks will be awarded for correct choice, 1 mark will be deducted for incorrect choice question and zero mark will be awarded for unattempted question.*
4. *For Section B questions, 4 marks will be awarded for correct answer and zero for unattempted and incorrect answer.*
5. *Any textual, printed or written material, mobile phones, calculator etc. are not allowed for the students appearing for the test.*
6. *All calculations / written work should be done in the rough sheet provided with Question Paper.*

Section A

Q. 1. Let $\vec{u} = \hat{i} - \hat{j} - 2\hat{k}, \vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \vec{v}.\vec{w} = 2$ and $\vec{v} \times \vec{w} = \vec{u} + \lambda\vec{v}$. Then $\vec{u}.\vec{w}$ is equal to

(A) 2 **(B)** $\frac{3}{2}$

(C) 1 **(D)** $-\frac{2}{3}$

Q. 2. $\lim_{t\to 0}\left(1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + \ldots + n^{\frac{1}{\sin^2 t}}\right)^{\sin^2 t}$ is equal to

(A) n^2 **(B)** $\frac{n(n+1)}{2}$

(C) n **(D)** $n^2 + n$

Q. 3. Let α be a root of the equation $(a - c) x^2 + (b - a)x (c - b) = 0$ where a, b, c are distinct real numbers such that the matrix $\begin{bmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}$ is singular.

Then, the value of $\frac{(a-c)^2}{(b-a)(c-b)} + \frac{(b-a)^2}{(a-c)(c-b)} + \frac{(c-b)^2}{(a-c)(b-a)}$ is

(A) 12 **(B)** 9

(C) 3 **(D)** 6

Q. 4. The area enclosed by the curves $y^2 + 4x = 4$ and $y - 2x = 2$ is:

(A) 9 **(B)** $\frac{22}{3}$

(C) $\frac{23}{3}$ **(D)** $\frac{25}{3}$

Q. 5. Let $p, q \in \mathbb{R}$ and $(1 - \sqrt{3}i)^{200} = 2^{199} (p + iq)$, $i = \sqrt{-1}$. Then $p + q + q^2$ and $p - q + q^2$ are roots of the equation.

(A) $x^2 - 4x - 1 = 0$ **(B)** $x^2 - 4x + 1 = 0$

(C) $x^2 + 4x - 1 = 0$ **(D)** $x^2 + 4x + 1 = 0$

Q. 6. Let N denote the number that turns up when a fair die is rolled. If the probability that the system of equations

$x + y + z = 1$

$2x + Ny + 2z = 2$

$3x + 3y + Nz = 3$

has unique solution is $\frac{k}{6}$, then the sum of value of k and all possible values of N is

(A) 21 **(B)** 18

(C) 20 **(D)** 19

Q. 7. For three positive integers p, q, r, $x^{pq^2} = y^{qr} = z^{p^2r}$ and $r = pq + 1$ such that 3, $3\log_y x$, $3 \log_z y$, $7 \log_x z$ are in A.P. with common difference $\frac{1}{2}$.

Then $r - p - q$ is equal to

(A) –6 **(B)** 12

(C) 6 **(D)** 2

Q. 8. The relation R = $\{(a, b): gcd\,(a, b) = 1, 2a \neq b, a, b \in Z\}$ is:

(A) reflexive but not symmetric

(B) transitive but not reflexive

(C) symmetric but not transitive

(D) neither symmetric nor transitive

Q. 9. Let PQR be a triangle. The points A, B and C are on the sides QR, RP and PQ respectively such that $\frac{QA}{AR}=\frac{RB}{BP}=\frac{PC}{CQ}=\frac{1}{2}$. Then $\frac{\text{Area}(\Delta PQR)}{\text{Area}(\Delta ABC)}$ is equal to

(A) 4 **(B)** 3
(C) 1 **(D)** 2

Q. 10. Let $y = y(x)$ be the solution of the differential equation $x^3dy + (xy - 1)\,dx = 0$, $x > 0$, $y\left(\frac{1}{2}\right) = 3 - e$. Then $y(1)$ is equal to

(A) 1 **(B)** e
(C) 3 **(D)** $2 - e$

Q. 11. If A and B are two non-zero $n \times n$ matrices such that $A^2 + B = A^2 B$, then

(A) $A^2 = I$ or $B = I$ **(B)** $A^2B = I$
(C) $AB = I$ **(D)** $A^2B = BA^2$

Q. 12. The equation $x^2 - 4x + [x] + 3 = x[x]$, where $[x]$ denotes the greatest integer function, has :

(A) a unique solution in $(-\infty, 1)$
(B) no solution
(C) exactly two solutions in $(-\infty, \infty)$
(D) a unique solution in $(-\infty, \infty)$

Q. 13. Let a tangent to the curve $y^2 = 24x$ meet the curve $xy = 2$ at the points A and B. Then the mid points of such line segment AB lie on a parabola with the

(A) length of latus rectum $\frac{3}{2}$
(B) directrix $4x = -3$
(C) length of latus rectum 2
(D) directrix $4x = 3$

Q. 14. Let Ω be the sample space and $A \subseteq \Omega$ be an event. Given below are two statements :
(S_1) : If $P(A) = 0$, then $A = \phi$
(S_2): If $P(A) = 1$, then $A = \Omega$
Then

(A) both (S_1) and (S_2) are true
(B) only (S_1) is true
(C) only (S_2) is true
(D) both (S_1) and (S_2) are false

Q. 15. The value of $\sum_{r=0}^{22} {}^{22}C_r \, {}^{23}C_r$ is

(A) ${}^{44}C_{23}$ **(B)** ${}^{45}C_{23}$
(C) ${}^{44}C_{22}$ **(D)** ${}^{45}C_{24}$

Q. 16. The distance of the point (–1, 9, –16) from the plane $2x + 3y - z = 5$ measured parallel to the line $\frac{x+4}{3} = \frac{2-y}{4} = \frac{z-3}{12}$ is

(A) 31 **(B)** $13\sqrt{2}$
(C) $20\sqrt{2}$ **(D)** 26

Q. 17. $\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$ is equal to:

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$ **(D)** $\frac{\pi}{2}$

Q. 18. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Then at $x = 0$

(A) f is continuous but not differentiable
(B) f and f' both are continuous
(C) f' is continuous but not differentiable
(D) f is continuous but f' is not continuous

Q. 19. The compound statement $(\sim(P \wedge Q)) \vee ((\sim P) \wedge Q) \Rightarrow ((\sim P) \wedge (\sim Q))$ is equivalent to

(A) $(\sim Q) \vee P$
(B) $((\sim P) \vee Q) \wedge (\sim Q)$
(C) $(\sim P) \vee Q$
(D) $((\sim P) \vee Q) \wedge ((\sim Q) \vee P)$

Q. 20. The distance of the point (7, −3, −4) from the plane passing through the points (2, −3,1), (−1,1, −2) and (3, −4,2) is :

(A) 5 **(B)** 4
(C) $5\sqrt{2}$ **(D)** $4\sqrt{2}$

Section B

Q. 21. Let $\lambda \in \mathbb{R}$ and let the equation E be $|x|^2 - 2|x| + |\lambda - 3| = 0$. Then the largest element in the set S $= \{x + \lambda : x$ is an integer solution of E$\}$ is

Q. 22. Let a tangent to the curve $9x^2 + 16y^2 = 144$ intersect the coordinate axes at the points A and B. Then, the minimum length of the line segment AB is

Q. 23. The shortest distance between the lines $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-6}{2}$ and $\frac{x-6}{3} = \frac{1-y}{2} = \frac{z+8}{0}$ is equal to

Q. 24. Suppose $\sum_{r=0}^{2023} r^2 \, {}^{2023}C_r = 2023 \times \alpha \times 2^{2022}$. Then the value of α is

Q. 25. The value of $\frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$ is

Q. 26. The number of 9 digit numbers, that can be formed using all the digits of the number 123412341 so that the even digits occupy only even places, is

Q. 27. A boy needs to select five courses from 12 available courses, out of which 5 courses are language courses. If he can choose at most two language courses, then the number of ways he can choose five courses is

Q. 28. The 4^{th} term of a GP is 500 and its common ratio is $\frac{1}{m}, m \in N$. Let S_n denote the sum of the first n terms of this GP. If $S_6 > S_5 + 1$ and $S_7 < S_6 + \frac{1}{2}$, then the number of possible values of m is

Q. 29. Let C be the largest circle centred at (2,0) and inscribed in the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$. If (1, α) lies on C, then 10 α^2 is equal to

Q. 30. The value of $12\int_0^3 \left|x^2-3x+2\right| dx$ is

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	C	Triple Products	Vector Algebra
2	C	Methods of Evaluation of Limits	Limits
3	C	Quadratic Equation and its Solution	Quadratic Equations
4	A	Area Bounded by Curves	Area under Curves
5	B	Euler's law	Complex Numbers
6	C	System of linear equations	Matrices and Determinants
7	D	Arithmetic Progressions	Sequences and Series
8	D	Equivalence Relations	Set Theory and Relations
9	B	Scalar and Vector Products	Vector Algebra
10	A	Solution of Linear Differential Equations	Differential Equations
11	D	Operations on Matrices	Matrices and Determinants
12	D	Quadratic Equations and its solution	Quadratic Equations
13	D	Interaction Between Two Conics	Hyperbola
14	A	Basics of Probability	Probability
15	B	Properties of Binomial Coefficients	Binomial Theorem
16	D	Intersection of a Line and a Plane	Three Dimensional Geometry
17	A	Basics of Inverse Trigonometric Functions	Inverse Trigonometric Functions
18	D	Differentiability of a Function	Continuity and Differentiability
19	C	Logical Operations	Mathematical Reasoning
20	C	Plane and a Point	Three Dimensional Geometry
21	[5]	Algebra of Functions	Function
22	[7]	Properties of Ellipse	Ellipse
23	[14]	Skew Lines	Three Dimensional Geometry
24	[1012]	Properties of Binomial Coefficients	Binomial Theorem
25	[2]	Properties of Definite Integrals	Definite Integration
26	[60]	Permutations	Permutations and Combinations
27	[546]	Combinations	Permutations and Combinations
28	[12]	Geometric Progressions	Sequences and Series
29	[118]	Interaction between Two Conics	Ellipse
30	[22]	Properties of Definite Integrals	Definite Integration

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Solutions

Section A

1. Option (C) is correct.

Given, $\vec{v} \times \vec{w} = \vec{u} + \lambda\vec{v}$

Taking dot product with $\vec{v}$

$\Rightarrow (\vec{v} \times \vec{w}).\vec{v} = (\vec{u} + \lambda\vec{v}).\vec{v}$

$\Rightarrow 0 = \vec{u}.\vec{v} + \lambda\vec{v}.\vec{v}$

$\Rightarrow 0 = (\hat{i} - \hat{j} - 2\hat{k}).(2\hat{i} + \hat{j} - \hat{k}) + \lambda |\vec{v}|^2$

$\Rightarrow 0 = 2 - 1 + 2 + \lambda (4 + 1 + 1)$

$\Rightarrow \lambda = \frac{-3}{6} = \frac{-1}{2}$

So, $\vec{v} \times \vec{w} = \vec{u} + \lambda\vec{v} = \vec{u} - \frac{1}{2}\vec{v}$

Taking dot product with $\vec{w}$

$\Rightarrow (\vec{v} \times \vec{w}).\vec{w} = \vec{u}.\vec{w} - \frac{1}{2}\vec{v}.\vec{w}$

$\Rightarrow 0 = \vec{u}.\vec{w} - \frac{1}{2}(2)$

$\Rightarrow \vec{u}.\vec{w} = 1$

HINT:

(1) $(\vec{a} \times \vec{b}).\vec{b} = 0$

(2) Take dot product to obtain the desired Dot - product

2. Option (C) is correct.

Let $l = \lim_{t\to 0}\left(1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + \ldots + n^{\frac{1}{\sin^2 t}}\right)^{\sin^2 t}$

$= \lim_{t\to 0} n\left[\left(\frac{1}{n}\right)^{\text{cosec}^2 t} + \left(\frac{2}{n}\right)^{\text{cosec}^2 t} + \ldots + \left(\frac{n}{n}\right)^{\text{cosec}^2 t}\right]^{\sin^2 t}$

$= n\lim_{t\to 0}\left[\sum_{r=1}^{n-1}\left(\frac{r}{n}\right)^{\text{cosec}^2 t} + 1\right]^{\sin^2 t} = n\lim_{t\to 0}\left[\sum_{r=1}^{n-1} 0 + 1\right]^{\sin^2 t}$

$\left[\text{as} \left(\frac{r}{n}\right)^{\text{cosec}^2 t} \to 0 \because 0 < \frac{r}{n} < 1 \,\&\, \text{cosec}^2 t \to \infty\right]$

$= n$

HINT:

(1) As $n \to \infty, \frac{r}{n} \to 0,$ for $0 < r < n$

(2) As $x \to 0$, $\text{cosec}^2 x \to \infty$

3. Option (C) is correct.

Let P = $\begin{bmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}$

Given that P is singular

$\Rightarrow |P| = 0$

$\Rightarrow \alpha^2 (c - b) - \alpha (c - a) + 1(b - a) = 0$

$\Rightarrow \alpha^2 (c - b) + \alpha (a - c) + (b - a) = 0$

Put $\alpha = 1$, $c - b + a - c + b - a = 0$

So, $\alpha = 1$ is a root

Now, $\sum \frac{(a-c)^2}{(b-a)(c-b)} = M$ (say)

$= \sum \frac{(a-c)^2}{(b-a)(c-b)}.\left(\frac{a-c}{a-c}\right) = \sum \frac{(a-c)^3}{(b-a)(c-b)(a-c)}$

$= \frac{(a-c)^3 + (c-b)^3 + (b-a)^3}{(b-a)(c-b)(a-c)}$

Here, $(a - c) + (c - b) + (b - a) = 0$

$\Rightarrow (a - c)^3 + (c - b)^3 + (b - a)^3 = 3 (a - c) (c - b) (b - a)$

$\therefore M = \frac{3(a-c)(c-b)(b-a)}{(b-a)(c-b)(a-c)}$

$\Rightarrow M = 3$

HINT:

(1) If $a + b + c = 0, a^3 + b^3 + c^3 = 3abc$

(2) $ax^2 + bx + c = 0$ with roots α, β

$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

4. Option (A) is correct.

Let $C_1 : y^2 + 4x = 4$ & $C_2 : y - 2x = 2$

Solving C_1 & C_2

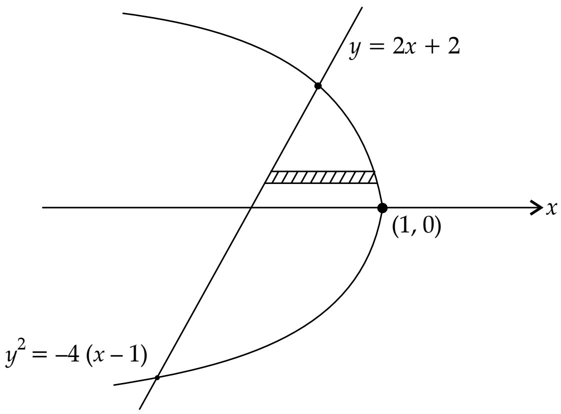
$\Rightarrow y^2 + 4\left(\frac{y}{2} - 1\right) = 4$

$\Rightarrow y^2 + 2y - 4 = 4$

$\Rightarrow y^2 + 2y - 8 = 0$

$\Rightarrow (y - 2)(y + 4) = 0$

$\Rightarrow y = -4, 2$



$$\text{Area} = \int (x_2 - x_1)dy$$

$$\Rightarrow \text{Area} = \int_{-4}^{2}\left\{\left(\frac{4-y^2}{4}\right)-\left(\frac{y-2}{2}\right)\right\}dy$$

$$\Rightarrow \text{Area} = \int_{-4}^{2}\left(2-\frac{y}{2}-\frac{y^2}{4}\right)dy$$

$$\Rightarrow \text{Area} = \left[2y-\frac{y^2}{4}-\frac{y^3}{12}\right]_{-4}^{2}$$

$$\Rightarrow \text{Area} = 2(2-(-4))-\frac{1}{4}(4-16)-\frac{1}{12}(8-(-64))$$

$\Rightarrow$ Area = 12 + 3 – 6

$\Rightarrow$ Area = 9 sq. units

HINT:

(1) Solve both the curves to get the point of intersection.

(2) Plot both the curves & think of a strip either horizontal or vertical and then integrate.

5. Option (B) is correct.

Given $2^{199}(p+iq) = (1-\sqrt{3}i)^{200}$

$$\Rightarrow \frac{2^{199}}{2^{200}}(p+iq) = \left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{200}$$

$$\Rightarrow p + iq = 2\left(\text{C is}\left(-\frac{\pi}{3}\right)\right)^{200}$$

$$\Rightarrow p + iq = 2\left(e^{-i\frac{\pi}{3}}\right)^{200}$$

$$\Rightarrow p + iq = 2e^{-i\frac{200\pi}{3}}$$

$$\Rightarrow p + iq = 2\left(\cos\left(\frac{200\pi}{3}\right)-i\sin\left(\frac{200\pi}{3}\right)\right)$$

$$\Rightarrow p + iq = 2\left[\cos\left(66\pi+\frac{2\pi}{3}\right)-i\sin\left(66\pi+\frac{2\pi}{3}\right)\right]$$

$$\Rightarrow p + iq = 2\left[\cos\left(\frac{2\pi}{3}\right)-i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$\Rightarrow p + iq = 2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow p = -1, q = -\sqrt{3}$$

Now, $p + q + q^2 = -1-\sqrt{3}+(-\sqrt{3})^2 = 2-\sqrt{3}$

$p - q + q^2 = -1-(-\sqrt{3})+(-\sqrt{3})^2 = 2+\sqrt{3}$

Equation whose roots are $p + q + q^2$ & $p - q + q^2$ having

Sum of roots (S) = $(2-\sqrt{3})+(2+\sqrt{3}) = 4$

and product of roots (P) = $(2-\sqrt{3})(2+\sqrt{3}) = 4-3 = 1$

Required quadratic equation is $x^2 - (S)\,x + P = 0$

$\Rightarrow x^2 - 4x + 1 = 0$

HINT:

(1) Use $\cos\theta + i\sin\theta = e^{i\theta}$

(2) Quadratic equation whose roots are α and β is

$$x^2 - (\alpha+\beta)\,x + \alpha\beta = 0$$

6. Option (C) is correct.

System of equations is,

$x + y + z = 1$

$2x + Ny + 2z = 2$

$3x + 3y + Nz = 3$

For unique solution, $\Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & N & 2 \\ 3 & 3 & N \end{vmatrix} \neq 0$$

$\Rightarrow 1\,(N^2 - 6) - (2N - 6) + (6 - 3N) \neq 0$

$\Rightarrow N^2 - 5N + 6 \neq 0$

$\Rightarrow (N - 2)\,(N - 3) \neq 0$

Therefore $N \neq 2, N \neq 3$

So, favourable cases are {1, 4, 5, 6}

Total cases ≡ {1, 2, 3, 4, 5, 6}

$$\text{Hence, probability} = \frac{\text{Number of favourable cases}}{\text{Total cases}}$$

$$\Rightarrow \frac{k}{6} = \frac{4}{6} \Rightarrow k = 4$$

So, sum of all possible values of k & N

= 4 + (1 + 4 + 5 + 6) = 20

HINT:

(1) For system of linear equations have unique solution $\Delta \neq 0$

(2) $\text{Probability} = \dfrac{\text{Number of favourable cases}}{\text{Total number of cases}}$

7. Option (D) is correct.

Given than, $x^{pq^2} = y^{qr} = z^{p^2 r}$ ...(1)

Also, 3, $3\log_y x$, $3\log_z y$, $7\log_x z$ are in A.P.

$\Rightarrow 3+\frac{1}{2}=3\log_y x$

$\Rightarrow \log_y x=\frac{7}{6}$

$\Rightarrow x = (y)^{7/6} \Rightarrow x^6 = y^7$

$3\log_z y = 3+2\left(\frac{1}{2}\right)=4$ ...(2)

$\Rightarrow \log_z y=\frac{4}{3}$

$\Rightarrow y = (z)^{4/3}$

$\Rightarrow y^3 = z^4$ ...(3)

$7\log_x(z) = 3+3\left(\frac{1}{2}\right)=\frac{9}{2}$

$\Rightarrow \log_x(z) = \frac{9}{14}$

$\Rightarrow z = (x)^{9/14}$

$\Rightarrow z^{14} = x^9$ ...(4)

Now from (1), we have

$$x^{pq^2} = \left(x^{\frac{6}{7}}\right)^{qr} = \left(x^{\frac{9}{14}}\right)^{p^2r}$$

$\Rightarrow pq^2 = \frac{6}{7}qr = \frac{9}{14}p^2r$

$\Rightarrow pq = \frac{6}{7}r, q^2 = \frac{9}{14}pr$

Also, $r = pq + 1$

$\Rightarrow r = \frac{6}{7}r+1$

$\Rightarrow \frac{r}{7}=1 \Rightarrow r = 7$

Now, $q^2 = \frac{9}{14}pr$

$\Rightarrow q(q^2) = \left(\frac{9}{14}r\right)(pq)$

$\Rightarrow q^3 = \left(\frac{9}{14}\right)r\left(\frac{6}{7}\right)r$

$\Rightarrow q^3 = \frac{9\times 6}{14\times 7}\times(7)^2$

$\Rightarrow q^3 = 27$

$\Rightarrow q = 3$

And, $pq = \frac{6}{7}r$

$\Rightarrow p = \frac{6}{7}\times\frac{7}{3}$

$\Rightarrow p = 2$

So, $r - p - q$

$= 7 - 2 - 3 = 2$

HINT:

(1) For an A.P., $T_n = a + (n-1)d$, where a & d are first term & common difference respectively.

(2) $\log_b(a) = c \Rightarrow a = b^c$

(3) If $a^b = a^q \Rightarrow p = q$

8. Option (D) is correct.

$R = \{(a, b) : gcd(a, b) = 1, 2a \neq b, a, b \in Z\}$

Reflexive:

Check for (a, a)

$gcd(a, a) = a$

So, R is not reflexive

Symmetric:

$(a, b) \in R \Rightarrow gcd\,(a, b) = 1$

Now check for (b, a)

$gcd\,(b, a) = gcd\,(a, b) = 1$

But $gcd\,(1, 2) = gcd\,(2, 1) = 1$

But $b \neq 2a$, So R is not symmetric.

Transitive:

Consider, $gcd(2, 3) = 1$

$\Rightarrow (2, 3) \in R$

Now, $gcd\,(3, 4) = 1$

$\Rightarrow (3, 4) \in R$

Again, $gcd\,(2, 4) = 2 \neq 1$

$\Rightarrow (2, 4) \notin R$

$\Rightarrow$ R is not Transitive.

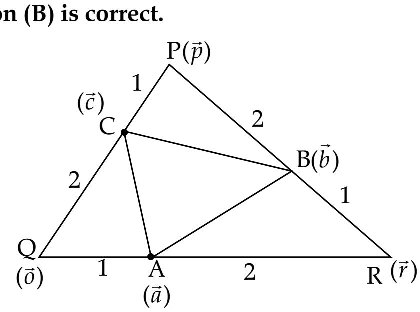
HINT:

(1) R is reflexive if $(a, a) \in R\ \forall\ a \in A$

(2) If $a, b \in A$, $(a, b) \in R \Rightarrow (b, a) \in R$, then R is symmetric.

(3) If $a, b, c \in A$, $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$, then R is transitive.

9. Option (B) is correct.



Let position vector of Q, P, and R be $\vec{o}, \vec{p}$ and $\vec{r}$ respectively.

Again, let position vector of points A, B, C be $\vec{a}, \vec{b}$ & $\vec{c}$ respectively.

Using section formula

$\vec{a} = \frac{\vec{r}}{3}, \vec{b} = \frac{\vec{p}+2\vec{r}}{3}, \vec{c} = \frac{2\vec{p}}{3}$

Area of $\Delta PQR = \left|\frac{1}{2}|\overrightarrow{QP}\times\overrightarrow{QR}|\right| = \frac{1}{2}|\vec{r}\times\vec{p}|$

Area of $\Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

$$= \frac{1}{2}\left|\left(\frac{\vec{r}}{3}\times\left(\frac{\vec{p}+2\vec{r}}{3}\right)\right)+\left(\frac{\vec{p}+2\vec{r}}{3}\right)\times\left(\frac{2\vec{p}}{3}\right)+\left(\frac{2\vec{p}}{3}\times\frac{\vec{r}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\frac{\vec{r}\times\vec{p}}{9}+4\left(\frac{\vec{r}\times\vec{p}}{9}\right)+\frac{2}{9}(\vec{p}\times\vec{r})\right|, \text{ as } \vec{r}\times\vec{r}=0$$

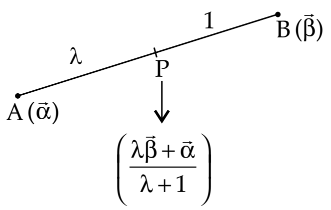
$$= \frac{1}{18}|\vec{r}\times\vec{p}+4(\vec{r}\times\vec{p})-2(\vec{r}\times\vec{p})| \text{ as } \vec{p}\times\vec{r}=-(\vec{r}\times\vec{p})$$

$$= \frac{|\vec{r}\times\vec{p}|}{6}$$

So, $\dfrac{\text{area of } \Delta PQR}{\text{area of } \Delta ABC} = 3$

HINT:

(1)



(2) Think of calculating the area of triangle using cross product.

10. Option (A) is correct.

$x^3 dy + (xy - 1)\,dx = 0$

Also, $y\left(\frac{1}{2}\right) = 3 - e \,\&\, x > 0$

Now, $x^3\frac{dy}{dx} + xy - 1 = 0$

$\Rightarrow x^3\frac{dy}{dx} = 1 - xy$

$\Rightarrow x^3\frac{dy}{dx} + xy = 1$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x^2}\right)y = \frac{1}{x^3}$

This is a linear differential equation

I.F. $= e^{\int\frac{1}{x^2}dx} = e^{\frac{-1}{x}}$

So, differential equation becomes,

$ye^{\frac{-1}{x}} = \int e^{\frac{-1}{x}}\cdot\frac{1}{x^3}dx$

Put $\frac{-1}{x} = t \Rightarrow \frac{1}{x^2}dx = dt$

R.H.S. $= \int -\underset{\text{I}}{t}\,\underset{\text{II}}{e^t}\,dt$

Integrating by parts

R.H.S. $= -\,[te^t - \int e^t dt] = -\,te^t + e^t + c$

So, solution of differential equation is

$ye^{\frac{-1}{x}} = -e^{\frac{-1}{x}}\left(\frac{-1}{x} - 1\right) + c$

$\Rightarrow y = \left(\frac{1}{x} + 1\right) + ce^{\frac{1}{x}}$

At $x = \frac{1}{2}, y = 3 - e$

$\Rightarrow 3 - e = \frac{1}{\left(\frac{1}{2}\right)} + 1 + ce^{\frac{1}{\left(\frac{1}{2}\right)}}$

$\Rightarrow 3 - e = (2 + 1) + ce^2$

$\Rightarrow ce^2 = -e$

$\Rightarrow c = \frac{-1}{e}$

So, $y = \left(\frac{1}{x} + 1\right) + \left(-\frac{1}{e}\right)e^{\frac{1}{x}}$

Now, at $x = 1$, $y = \left(\frac{1}{1} + 1\right) + \left(\frac{-1}{e}\right)e^{\frac{1}{1}}$

$\Rightarrow y = 2 - 1 = 1$

HINT:

(1) Convert given differential equation to linear differential equation & find its integrating factor to obtain the solution

(2) $\int \underset{\text{I}}{f(x)}\,\underset{\text{II}}{g(x)}\,dx = (\int g(x)dx)f(x) - \int f'(x)(\int g(x)dx)dx$

Using integration by parts

11. Option (D) is correct.

Given, $A^2 + B = A^2B$ ...(1)

$\Rightarrow A^2 - A^2B + B = 0$

$\Rightarrow A^2(I - B) - (-B + I - I) = 0$

$\Rightarrow A^2(I - B) - (I - B) = -I$

$\Rightarrow (A^2 - I)(I - B) = -I$

$\Rightarrow (I - A^2)(I - B) = I$

$\Rightarrow (I - B)(I - A^2) = I$

$\Rightarrow I - B - A^2 + BA^2 = I$

$\Rightarrow BA^2 - B - A^2 = 0$ ...(2)

(1) + (2) $\Rightarrow A^2B = BA^2$

HINT:

(1) $-(-A) = A$ **(2)** $A(BC^2) = ABC^2$

(3) $(A^2B)C = A^2BC$

12. Option (D) is correct.

$x^2 - 4x + [x] + 3 = x[x]$, where [.] = GIF

$\Rightarrow x^2 - 4x + 3 = x[x] - [x]$

$\Rightarrow (x - 1)(x - 3) = (x - 1)[x]$

$\Rightarrow (x - 1)(x - 3 - [x]) = 0$

$\Rightarrow x = 1, x - 3 = [x]$

$\Rightarrow x = 1, x - [x] = 3$

$\Rightarrow x = 1, \{x\} = 3$, where {.} = Fractional part function

But $\{x\} \in [0, 1)$

So, $\{x\} \neq 3$

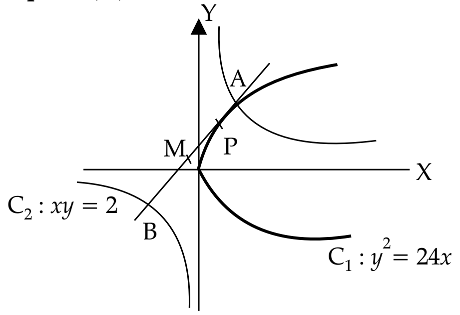
$\Rightarrow x = 1$

HINT:

(1) Take terms containing GIF on R.H.S. & factorize

(2) For $y = \{x\}$, where $\{.\}$ = FPF, $y \in [0, 1)$

13. Option (D) is correct.



$P \equiv (at^2, 2at)$, here $a = 6$

$\Rightarrow P \equiv (6t^2, 12t)$

Tangent to parabola at P(t)

$\Rightarrow ty = x + 6t^2$ ...(1)

Let M (h, k) be the mid-point of chord AB to hyperbola $xy = 2$

$AB \equiv \frac{x}{h} + \frac{y}{k} = 2$ ...(2)

Comparing (1) & (2), we get

$$\frac{-1}{\left(\frac{1}{h}\right)} = \frac{t}{\left(\frac{1}{k}\right)} = \frac{6t^2}{2}$$

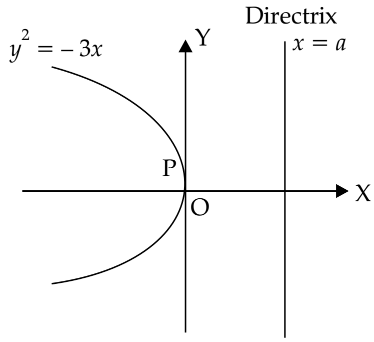
$\Rightarrow -h = tk = 3t^2$

$\Rightarrow h = -3t^2, k = 3t$

So, $h = -3\left(\frac{k}{3}\right)^2$

$\Rightarrow k^2 = -3h$

$\Rightarrow y^2 = -3x$



But $4a = 3$

$\Rightarrow a = \frac{3}{4}$

$\therefore$ Directrix is $x = \frac{3}{4}$

HINT:

(1) Write equation of tangent to parabola in parametric form.

(2) Write equation of chord to rectangular hyperbola $xy = c^2$ whose middle point is given as (h, k).

14. Option (A) is correct.

Let $\Omega \equiv \{1, 2, 3, 4, 5, 6\}$ i.e., they are outcome of throwing a dice.

Let A ≡ Getting a number 7

Now, $P(A) = \frac{\text{Favourable cases}}{\text{Total cases}} = \frac{0}{6} = 0$

But $A = \phi$

$\Rightarrow S_2$ is true.

B ≡ Getting a number < 7

$P(B) = \frac{\text{Favourable cases}}{\text{Total cases}}$

$\Rightarrow P(B) = \frac{6}{6} = 1$

Since, $B = \Omega$

$\Rightarrow S_2$ is true.

HINT:

(1) Think of a case that can never happen.

(2) Think of a case that will always happen.

15. Option (B) is correct.

Let $S = \sum_{r=0}^{22} ({}^{22}C_r)({}^{23}C_r)$

Consider, $(1 + x)^{22} = {}^{22}C_0 x^0 + {}^{22}C_1 x^1 + ... + {}^{22}C_{22} x^{22}$ ...(1)

Again, $(x + 1)^{23} = {}^{23}C_0 x^{23} + {}^{23}C_1 x^{22} + ... + {}^{23}C_{23} x^0$ ...(2)

(1) × (2) gives

$(1 + x)^{45} = ({}^{22}C_0 x^0 + {}^{22}C_1 x^1 + ... + {}^{22}C_{22} x^{22}) \times ({}^{23}C_0 x^{23} + {}^{23}C_1 x^{22} + ... + {}^{23}C_{23} x^0)$ ...(3)

Again, $S = \sum_{r=0}^{22} ({}^{22}C_{22-r})({}^{23}C_r)$,

Using ${}^nC_r = {}^nC_{n-r}$

$\Rightarrow S = ({}^{22}C_{22})({}^{23}C_0) + ({}^{22}C_{21})({}^{23}C_1) + ... + ({}^{22}C_0)({}^{23}C_{22})$

From (3), comparing coefficient of x^{23} on both sides

or $S = {}^{45}C_{23}$

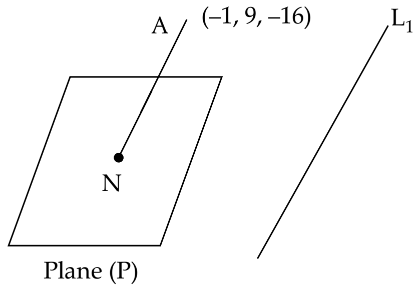
HINT:

(1) Use ${}^nC_r = {}^nC_{n-r}$

(2) Write expansion of $(1 + x)^n$ & $(x + 1)^n$. Multiply both series and compare coefficient of power of x to obtain the sum.

16. Option (D) is correct.

Plane (P) : $2x + 3y - z = 5$



where A ≡ (–1, 9, –16) & $L_1 \equiv \frac{x+4}{3} = \frac{y-2}{-4} = \frac{z-3}{12}$

Equation of line AN is,

$\frac{x-(-1)}{3} = \frac{y-9}{-4} = \frac{z-(-16)}{12} = \lambda$ (say)

$\Rightarrow x = 3\lambda - 1, y = -4\lambda + 9, z = 12\lambda - 16$

This point lies on plane (P)

$\Rightarrow 2(3\lambda - 1) + 3(-4\lambda + 9) - (12\lambda - 16) = 5$

$\Rightarrow -18\lambda + 41 = 5 \Rightarrow \lambda = 2$

So, N ≡ (3 (2) – 1, –4 (2) + 9, 12 (2) – 16)

⇒ N ≡ (5, 1, 8)

$AN = \sqrt{(5-(-1))^2 + (1-9)^2 + (8-(-16))^2}$

$= \sqrt{36+64+576} = \sqrt{676} = 26$

HINT:

(1) Write equation of line passing through (–1, 9, –16) & parallel to $\frac{x+4}{3} = \frac{2-y}{4} = \frac{z-3}{12}$

(2) Find point where it intersects the plane and then the required distance.

17. Option (A) is correct.

$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$

$= \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}(\sqrt{3}+1)}\right) + \sec^{-1}\left(\sqrt{\frac{4(2+\sqrt{3})}{3(2+\sqrt{3})}}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{4}{3}}\right)$

$= \frac{\pi}{6} + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$

HINT:

Take common & simplify the given expressions.

18. Option (D) is correct.

Given: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

As we know function is said to be continuous at a point if limiting value of the function at that point is equal to the functional value of the function at that point.

Now, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 = f(0)$

∴ $f(x)$ is continuous at $x = 0$.

As we know function is said to be differentiable at a point if LHD = RHD at that point.

Now, L.H.D. at $x = 0$, $f'(0^-) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{0-h}$

$= \lim_{h\to 0} \frac{-h^2 \sin\left(\frac{1}{h}\right) - 0}{-h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

Now, R.H.D. at $x = 0$, $f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

∵ L.H.D. = R.H.D. at $x = 0$

∴ $f(x)$ is differentiable at $x = 0$

Now, $f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

∵ limit of $f'(x) = 0$ oscillates

∴ $f'(x)$ is not continuous at $x = 0$

HINT:

(1) Function $f(x)$ is said to be continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$

(2) Function is said to be differentiable at the point if L.H.D. = R.H.D. at that point.

19. Option (C) is correct.

As we know $A \Rightarrow B \equiv \sim A \vee B$

So, $\sim(\sim P \wedge Q) \Rightarrow (\sim (P \vee Q)$

$= \sim[\sim(\sim P \wedge Q)] \vee (\sim P \vee Q)$

$= (\sim P \wedge Q) \vee (\sim P \vee Q)$

$= [\sim P \vee (\sim P \vee Q)] \wedge [Q \vee (\sim P \vee Q)]$

$= [\sim P \vee Q] \wedge [\sim P \vee Q] \equiv \sim P \vee Q$

HINT:

(1) Use $A \Rightarrow B = \sim A \vee B$

(2) Use $\sim (A \vee B) = \sim A \wedge \sim B$

20. Option (C) is correct.

As we know equation of plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \end{vmatrix} = 0$$

So, equation of plane passing through (2, –3, 1), (–1, 1, –2) and (3, –4, 2) is given by

$$\begin{vmatrix} x-2 & y+3 & z-1 \\ 3 & -4 & 3 \\ -1 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(4-3) - (y+3)(-3+3) + (z-1)(3-4) = 0$
$\Rightarrow x - 2 - z + 1 = 0$
$\Rightarrow x - z - 1 = 0$

Now, distance of the point (7, –3, –4) from plane $x - z - 1 = 0$ is

$$d = \left|\frac{7-(-4)-1}{\sqrt{2}}\right| \Rightarrow d = 5\sqrt{2}$$

HINT:

(1) Equation of plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

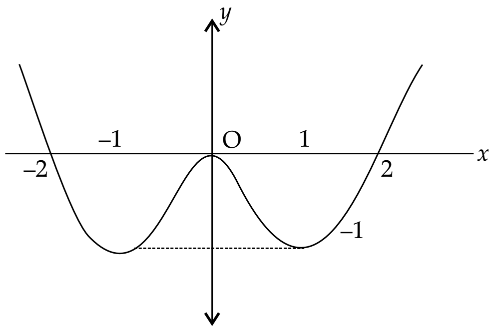
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \end{vmatrix} = 0$$

(2) Distance of the point (x_1, y_1, z_1) from plane $ax + by + cz + d = 0$ is given by

$$D = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

21. The correct answer is (5).

Given, E : $|x|^2 - 2|x| + |\lambda - 3| = 0$
S = $\{x + \lambda : x$ is an integer solution of E$\}$.
So, $|x|^2 - 2|x| = -|\lambda - 3|$
Lets draw the graph of $f(x) = |x|^2 - 2|x|$



It is clear from the figure, $-1 \le |x|^2 - 2|x| < \infty$ and $-|\lambda - 3| \le 0$
So, given equation holds only when $|\lambda - 3| \le 1$ and $x \in [-2, 2]$
$\Rightarrow -1 \le \lambda - 3 \le 1 \Rightarrow 2 \le \lambda \le 4$
For $x = 0, \lambda = 3$
For $x = \{-1, 1\}, \lambda = 4$ or 2
For $x = \{-2, 2\}, \lambda = 3$
So, largest element in the set S is 5

HINT:

(1) Write given equation as $|x|^2 - 2|x| = -|\lambda - 3|$ and draw the graph of $|x|^2 - 2|x|$ and analyse further using the concept of modulus function.

(2) Quadratic function $f(x) = x^2 + bx + c; a > 0$ has minimum value at $x = \frac{-b}{2a}$.

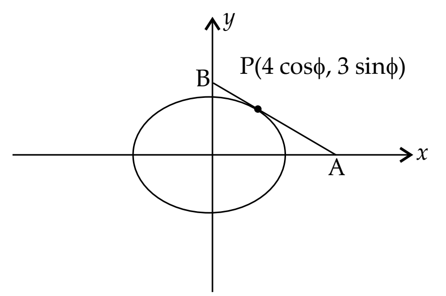
22. The correct answer is (7).

Given equation of curve is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

As we know equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$

So, equation of tangent to given ellipse at $(4\cos\phi, 3\sin\phi)$ is $\frac{x}{4}\cos\phi + \frac{y}{3}\sin\phi = 1$



So, coordinates of A = $(4\sec\phi, 0)$

Coordinates of B = $(0, 3\operatorname{cosec}\phi)$

Now, AB = $\sqrt{16\sec^2\phi + 9\operatorname{cosec}^2\phi}$

$\Rightarrow$ AB = $\sqrt{16(1+\tan^2\phi) + 9(1+\cot^2\phi)}$

$\Rightarrow$ AB = $\sqrt{25 + (4\tan\phi)^2 + (3\cot\phi)^2}$

$\Rightarrow$ AB = $\sqrt{25 + (4\tan\phi - 3\cot\phi)^2 + 24}$

$\Rightarrow (AB)_{min} = \sqrt{25+24} = 7$

HINT:

(1) Equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$

(2) Use $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

23. The correct answer is (14).

Given lines $L_1 : \frac{x-2}{3} = \frac{y+1}{2} = \frac{z-6}{2}$

$L_2 : \frac{x-6}{3} = \frac{1-y}{2} = \frac{z+8}{0}$

L_1 can be written as in vector from

$\vec{r} = (2\hat{i} - \hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 2\hat{k})$

L_2 can be written as in vector from

$\vec{r} = (6\hat{i} + \hat{j} - 8\hat{k}) + \mu(3\hat{i} - 2\hat{j})$

As we know shortest distance between two lines $\vec{r} = \vec{a} + \lambda\vec{p}$ and $\vec{r} = \vec{b} + \mu\vec{q}$ is given by

$$d = \left|\frac{(\vec{b} - \vec{a}).(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$$

Here, $\vec{a} = 2\hat{i} - \hat{j} + 6\hat{k}$,

$\vec{b} = 6\hat{i} + \hat{j} - 8\hat{k}$,

$\vec{p} = 3\hat{i} + 2\hat{j} + 2\hat{k}$,

$\vec{q} = 3\hat{i} - 2\hat{j}$

Now, $\vec{p} \times \vec{q} = \begin{vmatrix} i & j & k \\ 3 & 2 & 2 \\ 3 & -2 & 0 \end{vmatrix}$

$\Rightarrow \vec{p} \times \vec{q} = 4\hat{i} + 6\hat{j} - 12\hat{k} = 2(2\hat{i} + 3\hat{j} - 6\hat{k})$

Now, $\vec{b} - \vec{a} = 4\hat{i} + 2\hat{j} - 14\hat{k} = 2(2\hat{i} + \hat{j} - 7\hat{k})$

So, $(\vec{b} - \vec{a}).(\vec{p} \times \vec{q}) = 4[4 + 3 + 42] = 196$

And $|\vec{p} \times \vec{q}| = 2\sqrt{4 + 9 + 36} = 14$

$\therefore$ Shortest distance between given lines is $d = \left|\frac{196}{14}\right| = 14$

HINT:

(1) Write the given equation of line in vector form and use distance between two lines $\vec{r} = \vec{a} + \lambda\vec{p}$ and $\vec{r} = \vec{b} + \mu\vec{q}$ is given by $d = \left|\frac{(\vec{b} - \vec{a}).(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

(2) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

24. The correct answer is (1012).

Given: $\sum_{r=0}^{2023} r^2\, {}^{2023}C_r = 2023 \times \alpha \times 2^{2022}$

Let A = $\sum_{r=0}^{2023} r^2\, {}^{2023}C_r$

$\Rightarrow$ A = $\sum_{r=0}^{2023} r^2 \frac{2023}{r}\, {}^{2022}C_{r-1}$ $\quad \left\{\because {}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}\right\}$

$\Rightarrow$ A = $\sum_{r=1}^{2023} r(2023)\, {}^{2022}C_{r-1}$

$\Rightarrow$ A = $2023\left\{\sum_{r=1}^{2023} (r-1)\, {}^{2022}C_{r-1} + \sum_{r=1}^{2023} {}^{2022}C_{r-1}\right\}$

$\Rightarrow$ A = 2023 $\{2022\ 2^{2021} + 2^{2022}\}$

$\{\because {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + ... + n\,{}^nC_n = n\,2^{n-1}$ and ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n\}$

$\Rightarrow$ A = $2023 \times 2022 \times 2^{2021} + 2^{2022} \times 2023$

$\Rightarrow$ A = $2023 \times 2^{2022}(1011 + 1)$

$\Rightarrow$ A = $1012 \times 2023 \times 2^{2022}$

$\Rightarrow 2023 \times \alpha \times 2^{2022} = 1012 \times 2023 \times 2^{2022}$

$\Rightarrow \alpha = 1012$

HINT:

(1) Simplify given expression using ${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}$

(2) Use ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + ... + n\,{}^nC_n = n2^{n-1}$

(3) Use ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

25. The correct answer is (2).

Let I = $\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$ ...(1)

Using $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$\Rightarrow$ I = $\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\cos x)^{2023} + (\sin x)^{2023}} dx$ ...(2)

(1) + (2), 2I = $\int_0^{\frac{\pi}{2}} dx$

$\Rightarrow$ I = $\frac{1}{2}\left(\frac{\pi}{2} - 0\right) \Rightarrow$ I = $\frac{\pi}{4}$

So, $\frac{8}{\pi}(\text{I}) = \frac{8}{\pi}.\frac{\pi}{4} = 2$

HINT:

Use property: $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

26. The correct answer is (60).

$1 \rightarrow 3$ times

$2 \rightarrow 2$ times

$3 \rightarrow 2$ times

$4 \rightarrow 2$ times

_ X _ X _ X _ X _

O E O E O E O E O

O - odd, E - even

Number of ways for even digits to occupy even places

$= \frac{4!}{2!2!} = \frac{24}{2 \times 2} = 6$

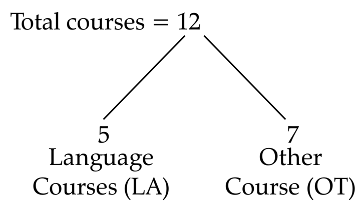
Total number of 9 digit numbers = $(6)\left(\frac{5!}{3!2!}\right)$

$= (6)\left(\frac{120}{6 \times 2}\right) = 60$

HINT:

Number of ways of arranging 'm' alike & 'n' distinct objects in a line are $\frac{(m+n)!}{m!}$

27. The correct answer is (546).



Number of ways

$= \{O(LA), 5\ (OT)\} + \{1(LA), 4\ (OT)\} + \{2\ (LA), 3\ (OT)\}$

$\Rightarrow ({}^5C_0)\ ({}^7C_5) + ({}^5C_1)\ ({}^7C_4) + ({}^5C_2)\ ({}^7C_3)$

$\Rightarrow (1)\ ({}^7C_2) + (5)\ ({}^7C_3) + \left(\frac{5\times 4}{2}\right)\ ({}^7C_3)$

$= \left(\frac{7\times 6}{2}\right) + (5)\left(\frac{7\times 6\times 5}{6}\right) + (10)\left(\frac{7\times 6\times 5}{6}\right)$

$= 21 + 175 + 350 = 546$

HINT:

(1) Make cases like (zero language, 5 other) or so on.
(2) Use ${}^nC_r = {}^nC_{n-r}$

28. The correct answer is (12).

Let first term of G.P. be 'a'.

Given, $T_4 = a\left(\frac{1}{m}\right)^3 = 500$

$\Rightarrow \frac{a}{m^3} = 500$

$\Rightarrow a = 500\ m^3$

Consider, $S_n - S_{n-1}$

$= a\left(\frac{1-r^n}{1-r}\right) - a\left(\frac{1-r^{n-1}}{1-r}\right)$, where $r = \frac{1}{m}$

$= \frac{a}{(1-r)}[1-r^n-1+r^{n-1}] = \frac{ar^{n-1}}{(1-r)}(1-r) = ar^{n-1}$

$\Rightarrow S_n - S_{n-1} = \frac{a}{m^{n-1}} \Rightarrow S_n - S_{n-1} = \frac{500m^3}{m^{n-1}}$

$\Rightarrow S_n - S_{n-1} = 500\ m^{4-n}$

Given, $S_6 - S_5 > 1$

$\Rightarrow 500\ m^{4-6} > 1$

$\Rightarrow \frac{500}{m^2} > 1$ ...(1)

Again, $S_7 - S_6 < \frac{1}{2}$

$\Rightarrow 500\ m^{4-7} < \frac{1}{2}$

$\Rightarrow \frac{500}{m^3} < \frac{1}{2}$ ...(2)

$(1) \Rightarrow m^2 < 500$

$(2) \Rightarrow m^3 > 1000$

So, $m \in \{11, 12, 13, ..., 22\}$

$\therefore$ Number of possible values of m is 12.

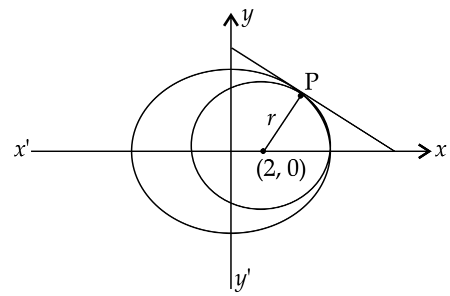
HINT:

(1) n^{th} term of a G.P. is $T_n = ar^{n-1}$, where 'a' and 'r' are first term and common ratio respectively.
(2) Sum of 'n' terms of a G.P.

$$S_n = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r < 1$$

29. The correct answer is (118).

Let ellipse be E: $\frac{x^2}{36} + \frac{y^2}{16} = 1$



Circle (C) $\equiv (x-2)^2 + (y-0)^2 = r^2$

For C to be largest possible circle, its radius has to be maximum.

Point P on ellipse $\equiv (6\cos\theta, 4\sin\theta)$

T = 0

Normal at P on ellipse should also be normal to circle as ellipse and circle are touching each other at P.

Normal $\equiv 6x\sec\theta - 4y\operatorname{cosec}\theta = 36 - 16$

It also passes through centre of circle i.e., (2, 0)

$\Rightarrow 12\sec\theta = 20$

$\Rightarrow \cos\theta = \frac{3}{5}$

So, $\sin\theta = \frac{4}{5}$

$P \equiv \left(6\times\frac{3}{5}, 4\times\frac{4}{5}\right) \equiv \left(\frac{18}{5}, \frac{16}{5}\right)$

Now, $r = \sqrt{\left(2-\frac{18}{5}\right)^2 + \left(0-\frac{16}{5}\right)^2}$

$\Rightarrow r = \sqrt{\frac{64}{25} + \frac{256}{25}}$

$\Rightarrow r = \frac{\sqrt{320}}{5} = \frac{8\sqrt{5}}{5} \Rightarrow r = \frac{8}{\sqrt{5}}$

Now $(1, \alpha)$ lies on c.

$\therefore \sqrt{(2-1)^2 + (0-\alpha)^2} = \frac{8}{\sqrt{5}}$

$1 + \alpha^2 = \frac{64}{5} \Rightarrow \alpha^2 = \frac{64}{5} - 1 = \frac{59}{5}$

$10\alpha^2 = \frac{59}{5} \times 10 = 118$

HINT:

(1) Think of common normal and remember that normal of circle passes through its centre.

(2) Normal to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P (a cosθ, b sinθ) is

$(a \sec\theta)\, x - (b\, \text{cosec}\theta)\, y = a^2 - b^2$

(3) Finally find radius of circle and make it equal to the distance between (1, α) and (2, 0).

30. The correct answer is (22).

Let $I = \int_0^3 |x^2 - 3x + 2|\, dx$

$$|x^2 - 3x + 2| = \begin{cases} x^2 - 3x + 2 & x \in (-\infty, 1] \cup [2, \infty) \\ -(x^2 - 3x + 2), & x \in (1, 2) \end{cases}$$

So, $I = \int_0^1 (x^2 - 3x + 2)dx - \int_1^2 (x^2 - 3x + 2)dx + \int_2^3 (x^2 - 3x + 2)dx$

$$\Rightarrow I = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2 + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$$

$$\Rightarrow I = \left[\frac{1}{3} - \frac{3}{2} + 2\right] - \left[\left(\frac{8}{3} - \frac{1}{3}\right) - \frac{3}{2}(4-1) + 2(2-1)\right] + \left[\left(\frac{27}{3} - \frac{8}{3}\right) - \frac{3}{2}(9-4) + 2(3-2)\right]$$

$$\Rightarrow I = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}$$

So, $12I = 12\left(\frac{11}{6}\right) = 22$

HINT:

(1) $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

(2) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$