JEE (Main) MATHEMATICS SOLVED PAPER

2023
08^{th} April Shift 2

Section A

Q. 1. Let

$A = \left\{\theta \in (0, 2\pi) : \frac{1+2i\sin\theta}{1-i\sin\theta} \text{ is purely imaginary}\right\}$.

Then the sum of the elements in A is

(1) π **(2)** 3π **(3)** 4π **(4)** 2π

Q. 2. Let P be the plane passing through the line $\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+5}{7}$ and the point (2, 4, –3). If the image of the point (–1, 3, 4) in the plane P is (α, β, γ) then $\alpha + \beta + \gamma$ is equal to

(1) 12 **(2)** 9 **(3)** 10 **(4)** 11

Q. 3. If $A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$. $A^{-1} = \alpha A + BI$ and $\alpha + \beta = -2$, then $4\alpha^2 + \beta^2 + \lambda^2$ is equal to :

(1) 14 **(2)** 12 **(3)** 19 **(4)** 10

Q. 4. The area of the quadrilateral ABCD with vertices A(2,1,1), B (1,2, 5), C(–2,–3, 5) and D (1, –6, –7) is equal to

(1) 54 **(2)** $9\sqrt{38}$ **(3)** 48 **(4)** $8\sqrt{38}$

Q. 5. $25^{190} - 19^{190} - 8^{190} + 2^{190}$ is divisible by

(1) 34 but not by 14 **(2)** 14 but not by 34

(3) Both 14 and 34 **(4)** Neither 14 nor 34

Q. 6. Let O be the origin and OP and OQ be the tangents to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ at the points P and Q on it. If the circumcircle of the triangle OPQ passes through the point $\left(\alpha, \frac{1}{2}\right)$, then a value of α is.

(1) $-\frac{1}{2}$ **(2)** $\frac{5}{2}$ **(3)** 1 **(4)** $\frac{3}{2}$

Q. 7. Let a_n be the n^{th} term of the series 5 + 8 + 14 + 23 + 35 + 50 +.... and $S_n = \sum_{k=1}^{n} a_k$. Then $S_{30} - a_{40}$ is equal to

(1) 11260 **(2)** 11280 **(3)** 11290 **(4)** 11310

Q. 8. If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and $\lim_{x \to \frac{1}{\alpha}} \left(\frac{1-\cos(x^2+bx+a)}{2(1-\alpha x)^2}\right)^{\frac{1}{2}} = \frac{1}{k}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)$. then k is equal to

(1) β **(2)** 2α **(3)** 2β **(4)** α

Q. 9. If the number of words, with or without meaning, which can be made using all the letters of the word MATHEMATICS in which C and S do not come together, is $(6!)k$, is equal to

(1) 1890 **(2)** 945 **(3)** 2835 **(4)** 5670

Q. 10. Let S be the set of all values of $\theta \in [-\pi, \pi]$ for which the system of linear equations

$x + y + \sqrt{3}z = 0$

$-x + (\tan\theta)y + \sqrt{7}z = 0$

$x + y + (\tan\theta) z = 0$

has non-trivial solution. Then $\frac{120}{\pi}\sum_{\theta \in S} \theta$ is equal to

(1) 20 **(2)** 40 **(3)** 30 **(4)** 10

Q. 11. For $a, b \in Z$ and $|a - b| \le 10$, let the angle between the plane P : $ax + y - z = b$ and the line l : $x - 1 = a - y = z + 1$ be $\cos^{-1}\left(\frac{1}{3}\right)$. If the distance of the point (6, –6, 4) from the plane P is $3\sqrt{6}$, then $a^4 + b^2$ is equal to

(1) 85 **(2)** 48 **(3)** 25 **(4)** 32

Q. 12. Let the vectors $\vec{u}_1 = \hat{i} + \hat{j} + a\hat{k}$, $\vec{u}_2 = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{u}_3 = c\hat{i} + \hat{j} + \hat{k}$ be coplanar. If the vectors $\vec{v}_1 = (a+b)\hat{i} + c\hat{j} + c\hat{k}$, $\vec{v}_2 = a\hat{i} + (b+c)\hat{j} + a\hat{k}$ and $\vec{v}_3 = b\hat{i} + b\hat{j} + (c+a)\hat{k}$ are also coplanar, then $6(a + b + c)$ is equal to

(1) 4 **(2)** 12 **(3)** 6 **(4)** 0

Q. 13. The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{11}$ is equal to

(1) $10^3 - 10$ **(2)** $11^3 - 11$

(3) $12^3 - 12$ **(4)** $13^3 - 13$

Q. 14. Let A = {1, 2, 3, 4, 5, 6, 7}. Then the relation $R = \{(x, y) \in A \times A : x + y = 7\}$ is

(1) Symmetric but neither reflexive nor transitive

(2) Transitive but neither symmetric nor reflexive

(3) An equivalence relation

(4) Reflexive but neither symmetric nor transitive

Q. 15. If the probability that the random variable X takes values x is given by $P(X=x) = k(x+1)3^{-x}$, $x = 0, 1, 2, 3, ...$, where k is a constant, then $P(X \ge 2)$ is equal to

(1) $\frac{7}{27}$ **(2)** $\frac{11}{18}$ **(3)** $\frac{7}{18}$ **(4)** $\frac{20}{27}$

Q. 16. The integral $\int \left(\left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x\right) \log_2 x \, dx$ is equal to

(1) $\left(\frac{x}{2}\right)^x \log_2\left(\frac{2}{x}\right) + C$ **(2)** $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

(3) $\left(\frac{x}{2}\right)^x \log_2\left(\frac{x}{2}\right) + C$ **(4)** $\left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x + C$

Q. 17. The value of $36 (4 \cos^2 9° - 1) (4 \cos^2 27° - 1) (4 \cos^2 81° - 1) (4 \cos^2 243° - 1)$ is

(1) 27 **(2)** 54 **(3)** 18 **(4)** 36

Q. 18. Let A (0, 1), B(1, 1) and C (1, 0) be the mid-points of the sides of a triangle with incentre at the point D. If the focus of the parabola $y^2 = 4ax$ passing through D is $(\alpha + \beta\sqrt{3}, 0)$, , where α and β are rational numbers, then $\frac{\alpha}{\beta^2}$ is equal to

(1) 6 **(2)** 8 **(3)** $\frac{9}{2}$ **(4)** 12

Q. 19. The negation of $(p \wedge (\sim q)) \vee (\sim p)$ is equivalent to

(1) $p \wedge (\sim q)$ **(2)** $p \wedge (q \wedge (\sim p))$

(3) $p \vee (q \vee (\sim p))$ **(4)** $p \wedge q$

Q. 20. Let the mean and variance of 12 observations be $\frac{9}{2}$ and 4 respectively. Later on, it was observed that two observations were considered as 9 and 10 instead of 7 and 14 respectively. If the correct variance is $\frac{m}{n}$, where m and n are coprime, then $m + n$ is equal to

(1) 316 **(2)** 317 **(3)** 315 **(4)** 314

Section B

Q. 21. Let R = $\{a, b, c, d, e\}$ and S = {1, 2, 3, 4}. Total number of onto functions $f : R \rightarrow S$ such that $f(a) \neq 1$ is equal to _____.

Q. 22. Let m and n be the numbers of real roots of the quadratic equations $x^2 - 12x + [x] + 31 = 0$ and $x^2 - 5[x + 2] - 4 = 0$ respectively, where $[x]$ denotes the greatest integer $\leq x$. Then $m^2 + mn + n^2$ is equal to ________.

Q. 23. Let P_1 be the plane $3x - y - 7z = 11$ and P_2 be the plane passing through the points (2, –1, 0), (2, 0, –1), and (5, 1, 1). If the foot of the perpendicular drawn from the point (7, 4, –1) on the line of intersection of the planes P_1 and P_2 is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to _____.

Q. 24. If domain of the function $\log_e\left(\frac{6x^2 + 5x + 1}{2x - 1}\right) + \cos^{-1}\left(\frac{2x^2 - 3x + 4}{3x - 5}\right)$ is $(\alpha, \beta) \cup (\gamma, \delta]$, then, $18 (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ is equal to

Q. 25. Let the area enclosed by the lines $x + y = 2$, $y = 0, x = 0$ and the curve $f(x) = \min\left\{x^2 + \frac{3}{4}, 1 + [x]\right\}$ where $[x]$ deontes the greatest integer $\leq x$, be A. Then the value of 12A is ______.

Q. 26. Let $0 < z < y < x$ be three real numbers such that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in an arithmetic progression and $x, \sqrt{2}y, z$ are in a geometric progression. If $xy + yz + zx = \frac{3}{\sqrt{2}}xyz$, then $3 (x + y + z)^2$ is equal to _________.

Q. 27. Let the solution curve $x = x(y), 0 < y < \frac{\pi}{2}$, of the differential equation $(\log_e (\cos y))^2 \cos y \, dx - (1 + 3x \log_e (\cos y)) \sin y \, dy = 0$ satisfy $x\left(\frac{\pi}{3}\right) = \frac{1}{2\log_e 2}$. If $x\left(\frac{\pi}{6}\right) = \frac{1}{\log_e m - \log_e n}$, where m and n are co-prime, then mn is equal to

Q. 28. Let $[t]$ denote the greatest integer function. If $\int_0^{24} [x^2] dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5}$, then $\alpha + \beta + \gamma + \delta$ is equal to _________.

Q. 29. The ordinates of the points P and Q on the parabola with focus (3.0) and directrix $x = -3$ are in the ratio 3 : 1. If R (α, β) is the point of intersection of the tangents to the parabola at P and Q, then $\frac{\beta^2}{\alpha}$ is equal to ________.

Q. 30. Let k and m be positive real numbers such that the function $f(x) = \begin{cases} 3x^2 + k\sqrt{x+1}, & 0 < x < 1 \\ mx^2 + k^2 & x \geq 1 \end{cases}$ is differentiable for all $x > 0$. Then $\frac{8f'(8)}{f'\left(\frac{1}{8}\right)}$ is equal to _____.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(3)	General form	Complex Numbers
2	(3)	Equation of plane	Three Dimensional Geometry
3	(1)	Charactersictic equation	Matrices and Determinants
4	(4)	Area of quadrilateral	Vector Algebra
5	(1)	Remainder theorem	Binomial Theorem
6	(2)	Circumcircle	Circle
7	(3)	Special series	Sequences and Series
8	(2)	Limits of trigonometry	Limits

Q. No.	Answer	Topic Name	Chapter Name
9	(4)	Number of words	Permutation and Combination
10	(1)	System of linear equations	Matrices and Determinants
11	(4)	Distance of a point from a plane	Three Dimensional Geometry
12	(2)	Scalar triple product	Vector Algebra
13	(3)	General term	Binomial Theorem
14	(1)	Equivalence relation	Relation and Function
15	(1)	Probaility distribuution	Probability
16	(4)	Indefinite Integral	Integral Calculus
17	(4)	Trigonometric relations	Trigonometry
18	(2)	Incentre of triangle	Parabola
19	(4)	Equivalent statement	Mathematical Reasoning
20	(2)	Mean, Variance	Statistics
21	[180]	Number of onto fuctions	Relation and Function
22	[9]	Roots of equation	Quadratic equations
23	[11]	Equation of plane	Three dimensional geometry
24	[20]	Domain of a function	Function
25	[17]	Area between the curves	Integral Calculus
26	[150]	A.P., G.P.	Sequences and series
27	[12]	Linear differential equation	Differential equations
28	[6]	Definite Integral	Integral Calculus
29	[16]	Parabola	Conic Section
30	[309]	First derivative	Differentiability

Solutions

Section A

1. Option (3) is correct.

Here, $z = \dfrac{1+2i\sin\theta}{1-i\sin\theta}\times\dfrac{1+i\sin\theta}{1+i\sin\theta}$

$$\frac{1+i\sin\theta+2i\sin\theta-2\sin^2\theta}{1-i^2\sin^2\theta}$$

$$=\frac{(1-2\sin^2\theta)+i(3\sin\theta)}{1+\sin^2\theta}$$

$\because$ z is purely imaginary, so Re $z = 0$

$$\Rightarrow \frac{1-2\sin^2\theta}{1+\sin^2\theta}=0$$

$$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}$$

$$\therefore A=\left[\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right] \qquad \because \theta\in(0,2\pi)$$

$$\therefore \text{Sum} = \frac{\pi+3\pi+5\pi+7\pi}{4} = \frac{16\pi}{4}=4\pi$$

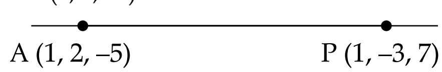
HINT:

For a complex number, $z = a + ib$, if z is purely imaginary, then Re $z = 0 \Rightarrow a = 0$

2. Option (3) is correct.

Equation of line : $\dfrac{x-1}{1}=\dfrac{y-2}{-3}=\dfrac{z+5}{7}$

Let B ≡ (2, 4, –3)



So, $\overrightarrow{AB} = (2-1)\hat{i}+(4-2)\hat{j}+(-3+5)\hat{k}$

$$=\hat{i}+2\hat{j}+2\hat{k}$$

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 7\\ 1 & 2 & 2\end{vmatrix} = (-6-14)\hat{i}-(2-7)\hat{j}+(2+3)\hat{k}$$

$$=-5(4\hat{i}-\hat{j}-\hat{k})$$

$\therefore$ Eqn. of plane is :

$4(x-1) + (-1)(y-2) - 1(z+5) = 0$

$\Rightarrow 4x - 4 - y + 2 - z - 5 = 0$

$\Rightarrow 4x - y - z - 7 = 0$

$\because$ Image of point (–1, 3, 4) is (α, β, γ)

So, $\dfrac{\alpha+1}{4}=\dfrac{\beta-3}{-1}=\dfrac{\gamma-4}{-1}=\dfrac{-2(-4-3-4-7)}{16+1+1}=2$

$\Rightarrow \alpha = 7, \beta = 1, \gamma = 2$

So, $\alpha+\beta+\gamma = 10$

HINT:

Equation of plane passing through the line and a point can be find by using the normal vector.

3. **Option (1) is correct.**

$A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$

$\Rightarrow |A - xI| = 0$

$\Rightarrow \begin{vmatrix} 1-x & 5 \\ \lambda & 10-x \end{vmatrix} = 0$

$\Rightarrow 10 - 11x + x^2 - 5\lambda = 0$

Also, $\Rightarrow A^{-1} = \alpha A + \beta I$

$\Rightarrow \alpha A^2 + \beta A - I = 0$

and $A^2 - 11A + (10 - 5\lambda) I = 0$

On solving, we get

$\alpha = \frac{1}{5}, \beta = -\frac{11}{5}$

So, $5\lambda - 10 = 5 \Rightarrow \lambda = 3$

$\therefore 4\alpha^2 + \beta^2 + \lambda^2$

$= 4\left(\frac{1}{25}\right) + \left(\frac{121}{25}\right) + 9$

$= \frac{125}{25} + 9 = 14$

HINT:

The characteristic equation is :

$|A - xI| = 0$

4. **Option (4) is correct.**

Here $\overline{AC} = (-2-2)\hat{i} + (-3-1)\hat{j} + (5-1)\hat{k}$

$= -4\hat{i} - 4\hat{j} + 4\hat{k}$

$\overline{BD} = (1-1)\hat{i} + (-6-2)\hat{j} + (-7-5)\hat{k}$

$= -8\hat{j} - 12\hat{k}$

So, area of quadrilateral $= \frac{1}{2}||\overline{AC} \times \overline{BD}|$

$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -4 & 4 \\ 0 & -8 & -12 \end{matrix}\right\|$

$= 8\sqrt{38}$ sq units.

HINT:

Area of quadrilateral = Half of product of diagonal vectors.

5. **Option (1) is correct.**

The given expression is divisible by 6 and 17.

Also, $25^{190} - 8^{190}$ is not divisible by 7

but $19^{190} - 2^{190}$ is divisible by 7,

So, $25^{190} - 19^{190} - 8^{190} + 2^{190}$ is divisible by 34 but not by 14.

6. **Option (2) is correct.**

Centre (3, –2)

Equation of circumcircle is

$x(x-3) + y(y+2) = 0$

$\Rightarrow x^2 - 3x + y^2 + 2y = 0$

Since $\left(\alpha, \frac{1}{2}\right)$ is on the circle

So $\alpha^2 - 3\alpha + \frac{1}{4} + 1 = 0$

$\Rightarrow 4\alpha^2 - 12\alpha + 5 = 0$

$\Rightarrow \alpha = \frac{12 \pm \sqrt{144-80}}{8}$

$\alpha = \frac{20}{8}, \frac{4}{8} = \frac{5}{2}, \frac{1}{2}$

HINT:

Equation of circumcircle whose diametric points are (a, b) & (c, d) is $(x-a)(x-c) + (y-b)(y-d) = 0$

7. **Option (3) is correct.**

Let $S_n = 5 + 8 + 14 + 23 + ... + a_n$

and $S_n = 0 + 5 + 8 + 14 + ... + a_n$

On subtracting, we get

$0 = 5 + 3 + 6 .... - a_n$

$\Rightarrow a_n = 5 + 3 + 6 + 9 + ... (n-1)$ terms

$= 5 + \left[\frac{(n-1)}{2}(6 + (n-2)3)\right]$

$= \frac{10 + 3n^2 - 3n}{2}$

So, $a_{40} = \frac{3(40)^2 - 3(40) + 10}{2}$

$= \frac{4800 - 120 + 10}{2} = 2345$

Now, $S_n = \sum_{k=1}^{n} a_k$

$\Rightarrow S_{30} = \frac{3\sum_{n=1}^{30} n^2 - 3\sum_{n=1}^{30} n + 10\sum_{n=1}^{30} 1}{2}$

$= 13635$

$\therefore S_{30} - a_{40} = 13635 - 2345 = 11290$

8. **Option (2) is correct.**

Since, α, β are roots of $ax^2 + bx + 1 = 0$

Replace $x \to \frac{1}{x}$

$\frac{a}{x^2} + \frac{b}{x} + 1 = 0 \Rightarrow x^2 + bx + a = 0$

So, $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots

Now, $\lim_{x \to \frac{1}{\alpha}} \left[\frac{1 - \cos(x^2 + bx + a)}{2(1 - \alpha x)^2}\right]^{\frac{1}{2}}$

$$= \lim_{x\to\frac{1}{\alpha}}\left[\frac{2\sin^2\frac{\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2}}{4\times 2\alpha^2\frac{\left(x-\frac{1}{\alpha}\right)^2\left(x-\frac{1}{\beta}\right)^2}{4}}\left(x-\frac{1}{\beta}\right)^2\right]^{\frac{1}{2}}$$

$$= \lim_{x\to\frac{1}{\alpha}}\left[\pm\frac{1}{2}\frac{\sin\frac{\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2}}{\alpha\frac{\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2}}\left(x-\frac{1}{\beta}\right)\right]$$

$$= \frac{1}{2\alpha}\left(\frac{-1}{\alpha}+\frac{1}{\beta}\right)$$

$$\Rightarrow \frac{1}{k}\left[\frac{1}{\beta}-\frac{1}{\alpha}\right] = \frac{1}{2\alpha}\left[\frac{1}{\beta}-\frac{1}{\alpha}\right]$$

$\Rightarrow k = 2\alpha$

9. Option (4) is correct.

Total number of words $= \frac{11!}{2!2!2!}$

Number of words in which C and S are together

$= \frac{10!}{2!2!2!}\times 2!$

So, required number of words

$= \frac{11!}{2!2!2!} - \frac{10!}{2!2!}$

$= \frac{11\times 10!}{2!2!2!} - \frac{10!}{2!2!}$

$= \frac{10!}{2!2!}\left[\frac{11}{2}-1\right] = \frac{10!}{2!2!}\times\frac{9}{2}$

$= 5670 \times 6!$

$\Rightarrow k\,(6!) = 5670 \times 6!$

$\Rightarrow k = 5670$

10. Option (1) is correct.

Since, the given system has a non trivial solution,

So, $\Delta = 0$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & \sqrt{3} \\ -1 & \tan\theta & \sqrt{7} \\ 1 & 1 & \tan\theta \end{vmatrix} = 0$$

$\Rightarrow 1(\tan^2\theta - \sqrt{7}) - 1(-\tan\theta - \sqrt{7}) + \sqrt{3}(-1-\tan\theta) = 0$

$\Rightarrow \tan^2\theta - \sqrt{7} + \tan\theta + \sqrt{7} - \sqrt{3} - \sqrt{3}\tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - \sqrt{3}) + 1(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow \tan\theta = \sqrt{3}$ or $\tan\theta = -1$

$\therefore \theta = \left\{\frac{\pi}{3}, \frac{-2\pi}{3}, \frac{-\pi}{4}, \frac{3\pi}{4}\right\}$

So, $\frac{120}{\pi}\sum_{\theta\in S}\theta = \frac{120}{\pi}\left\{\frac{4\pi - 8\pi - 3\pi + 9\pi}{12}\right\}$

$= \frac{120}{\pi}\left[\frac{2\pi}{12}\right] = 20$

HINT:

For a system of linear equation having non trivial solution, $\Delta = 0$

11. Option (4) is correct.

We have, $\theta = \cos^{-1}\frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{3}$

$\therefore \sin\theta = \sqrt{1-\left(\frac{1}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

The given plane line and are

$ax + y - z = b$ & $x - 1 = a - y = z + 1$

$\therefore \sin\theta = \frac{a.1 + (1)(-1) + (-1)(1)}{\sqrt{a^2+1^2+1^2}\sqrt{1^2+1^2+1^2}}$

$\Rightarrow \frac{a-1-1}{\sqrt{a^2+2}\sqrt{3}} = \frac{2\sqrt{2}}{3}$

$\Rightarrow 3(a-2) = 2\sqrt{6}\sqrt{a^2+2}$

$\Rightarrow 9\,(a^2 + 4 - 4a) = 24\,(a^2 + 2)$

$\Rightarrow 9a^2 + 36 - 36a = 24a^2 + 48$

$\Rightarrow a = \frac{-2}{5}, -2$

So, $a = -2$ $\because a \in Z$

Hence, the eqn. of plane is $-2x + y - z - b = 0$

Now, $d = \left|\frac{-12-6-4-b}{\sqrt{4+1+1}}\right| = 3\sqrt{6}$

$\Rightarrow |-(b + 22)| = 18$

$\Rightarrow b = 18 - 22 = -4$

$\therefore a^4 + b^2 = (-2)^4 + (-4)^2$

$= 16 + 16 = 32$

HINT:

Distance of a point (a_1, b_1, c_1) from the plane $ax + by + cz + d = 0$ is $d = \left|\frac{aa_1 + bb_1 + cc_1 + d}{\sqrt{a^2+b^2+c^2}}\right|$

12. Option (2) is correct.

Since, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are coplanar.

So, $[\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & a \\ 1 & b & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow a + b + c - 2 = abc$ ...(i)

Also, $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = 0$

$$\Rightarrow \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 0$$

$\Rightarrow 4abc = 0 \Rightarrow abc = 0$ ...(ii)

So, $a + b + c - 2 = 0$ [from (i)]

$\Rightarrow a + b + c = 2$

$\Rightarrow 6\,(a + b + c) = 12$

> **HINT:**
> If three non-zero vectors are coplanar, then their scalar triple product is zero.

13. Option (3) is correct.

General term of $\left(2x^2+\frac{1}{2x}\right)^{11}$ is:

$T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(\frac{1}{2x}\right)^r$

$= {}^{11}C_r\,2^{11-r}\,x^{22-2r}\,2^{-r}\,x^{-r}$

$= {}^{11}C_r\,2^{11-r}\,x^{22-3r}$

Now, $22-2r = 10$ and $22-3r = 7$

$\Rightarrow 3r = 12$ $\quad\Rightarrow 3r = 15$

$\Rightarrow r = 4$ $\quad\Rightarrow r = 5$

$\therefore$ Coeff. of $x^{10} = {}^{11}C_4 . 2^{11-8} = {}^{11}C_4 \times 8$

Coeff. of $x^7 = {}^{11}C_5 . 2^{11-10} = {}^{11}C_4 \times 2$

Now, required difference

$= {}^{11}C_4 \times 8 - {}^{11}C_5 \times 2$

$= \frac{11\times10\times9\times8\times7!}{4!\times7!}\times8-\frac{11\times10\times9\times8\times7\times6!\times2}{5!\,6!}$

$= \frac{11\times10\times9\times8\times8}{24}-\frac{11\times10\times9\times8\times7\times2}{120}$

$= 11\times10\times8\times3-11\times3\times4\times7$

$= 11\times3\times4\,[20-7]$

$= 11\times12\times13 = (12-1)\times12\times(12+1)$

$= 12\,(12^2-1) = 12^3-12$

14. Option (1) is correct.

Here, $A = \{1, 2, 3, 4, 5, 6, 7\}$

Since, $x + y = 7 \Rightarrow y = 7 - x$

So, $R = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$\because (a, b) \in R \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric only.

> **HINT:**
> For a relation,
> if $(a, a) \in R \Rightarrow$ R is reflexive
> if $(a, b) \in R \Rightarrow (b, a) \in R$ So, R is symmetric
> if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
> So, R is transitive

15. Option (1) is correct.

As, we know that sum of all the probabilities = 1

So, $\sum_{x=1}^{\infty} P(X=x)=1$

$\Rightarrow k[1 + 2.3^{-1} + 3.3^{-2} + ...\infty] = 1$

Let $S = 1+\frac{2}{3}+\frac{3}{3^2}+....+\infty$

$\Rightarrow \frac{S}{3}=0+\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+.....+\infty$

On subtracting, we get

$\frac{2S}{3}=1+\frac{1}{3}+\frac{1}{3^2}+....+\infty$

$\Rightarrow \frac{2S}{3}=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}$

$\Rightarrow \frac{2S}{3}=\frac{3}{2} \Rightarrow S=\frac{9}{4}$

So, $k\times\frac{9}{4}=1\Rightarrow k=\frac{4}{9}$

Now, $P(X \ge 2) = 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$=1-\frac{4}{9}(1)-\frac{4}{9}\times\frac{2}{3}$

$=1-\frac{4}{9}-\frac{8}{27}=\frac{27-12-8}{27}=\frac{7}{27}$

Sum of probabilities = 1

$\sum_{x=0}^{\infty} P(X=x)=1$

16. Option (4) is correct.

Note: Given integral is wrong it may be

$\int\left[\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x\right]\ln\left(\frac{ex}{2}\right)dx$

Let $I = \int\left[\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x\right]\ln\left(\frac{ex}{2}\right)dx$

$=\int\left[e^{x\ln x-x\ln 2}+e^{x\ln 2-x\ln x}\right]dx$

Let $x \ln x - x \ln 2 = t$

$(\ln x + 1 - \ln 2)dx = dt$

$\Rightarrow \ln\left(\frac{ex}{2}\right)dx=dt$

$\therefore I=\int\left[e^t-e^{-t}\right]dt$

$= e^t + e^{-t} + c$

$=\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x+c$

17. Option (4) is correct.

$4\cos^2\theta - 1 = 4\,(1 - \sin^2\theta) - 1$

$= 3 - 4\sin^2\theta$

$=\frac{3\sin\theta-4\sin^3\theta}{\sin\theta}=\frac{\sin 3\theta}{\sin\theta}$

So, $36\,(4\cos^2 9^\circ - 1)\,(4\cos^2 27^\circ - 1)\,(4\cos^2 81^\circ - 1)\,(4\cos^2 243^\circ - 1)$

$=36\left[\frac{\sin 27^\circ}{\sin 9^\circ}\times\frac{\sin 81^\circ}{\sin 27^\circ}\times\frac{\sin 243^\circ}{\sin 81^\circ}\times\frac{\sin 729^\circ}{\sin 243^\circ}\right]$

$=36\left[\frac{\sin 729^\circ}{\sin 9^\circ}\right]=36\times1=36$

18. Option (2) is correct.

So, $D\equiv\left(\frac{4}{2+2+2\sqrt{2}},\frac{4}{2+2+2\sqrt{2}}\right)$

$\equiv\left(\frac{2}{2+\sqrt{2}},\frac{2}{2+\sqrt{2}}\right)$

$$=\left(\frac{2}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}},\frac{2}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)$$

$\equiv (2-\sqrt{2}, 2-\sqrt{2})$

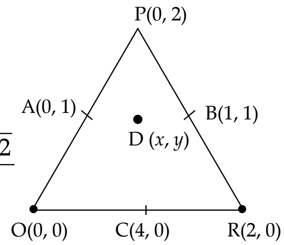
$\because y^2 = 4ax$

$(2-\sqrt{2})^2 = 4a(2-\sqrt{2})$

$\Rightarrow 4a = 2-\sqrt{2} \Rightarrow a = \frac{2-\sqrt{2}}{4}$

$\Rightarrow \frac{1}{2}-\frac{\sqrt{2}}{4} = \alpha+\beta\sqrt{2}$

$\Rightarrow \alpha = \frac{1}{2}, \beta = \frac{-1}{4}$ So, $\frac{\alpha}{\beta^2} = \frac{\frac{1}{2}}{\frac{1}{16}} = 8$



HINT:

The incentre of a triangle is the intersection point of all the three interior angle bisectors of the triangle.

19. Option (4) is correct.

$(p \wedge (\sim q)) \vee (\sim p)$

$\equiv (p \vee \sim p) \wedge (\sim q \vee \sim p)$

$\equiv T \wedge (\sim q \vee \sim p)$

$\equiv \sim q \vee \sim p$ negation $p \wedge q$

HINT:

$a \vee \sim a \equiv T$

$\sim a \vee \sim b \equiv b \wedge a$

20. Option (2) is correct.

Since, Mean $= \frac{9}{2}$

$\Rightarrow \Sigma x = \frac{9}{2}\times 12 = 54$

Also, variance = 4

$\Rightarrow \frac{\sum x^2}{12} = \left[\frac{\sum x_i}{12}\right]^2 = 4$

$\Rightarrow \frac{\sum x^2}{12} = 4+\frac{81}{4} = \frac{97}{4}$

$\Rightarrow \sum x^2 = 291$

$\sum x' = 54-(9+10)+7+14$

$= 54 - 19 + 21 = 56$

and $\sum x^2 = 291 - (81 + 100) + 49 + 196$

$= 291 - 181 + 49 + 196 = 355$

So, $\sigma^2_{new} = \frac{\sum x^2_{new}}{12} - \left(\frac{\sum x_{new}}{12}\right)^2$

$= \frac{355}{12} - \left(\frac{56}{12}\right)^2$

$= \frac{4260-3136}{144} = \frac{1124}{144} = \frac{281}{36}$

$= \frac{m}{n}$

$\Rightarrow m = 281$ & $n = 36$

$\Rightarrow m + n = 281 + 36 = 317$

HINT:

Mean $= \frac{\sum x}{n}$

Variance $(\sigma^2) = \frac{\sum x^2}{n} - \left[\frac{\sum x}{n}\right]^2$

Section B

21. Correct answer is [180].

Total number of onto functions

$= \frac{5!}{3!2!}\times 4! = \frac{5\times 4}{2}\times 24 = 240$

When $f(a) = 1$, number of onto functions

$= 4! + \frac{4!}{2!2!}\times 3! = 24 + 36 = 60$

So, required number of onto functions

$= 240 - 60 = 180$

22. Correct answer is [9].

The given eqn is : $x^2 - 12x + [x] + 31 = 0$

$\Rightarrow \{x\} - x = x^2 - 12x + 31$

$\Rightarrow \{x\} = x^2 - 11x + 31$

So, $0 \le x^2 - 11x + 31 < 1$

$\Rightarrow x^2 - 11x + 30 \le 0$

$\Rightarrow x \in (5, 6)$

$\therefore [x] = 5$

$\therefore x^2 - 12x + 5 + 31 = 0$

$\Rightarrow (x-6)^2 = 0 \Rightarrow x = 6$

Hence, $x \in \phi$ $\quad (\because x \in (5, 6))$

$\therefore m = 0$

Another equation is $x^2 - 5[x + 2] - 4 = 0$

Case I: $x \ge -2$

$x^2 - 5x - 14 = 0 \Rightarrow x = 7, -2$

Case II: $x < -2$

$x^2 + 5x + 6 = 0 \Rightarrow x = -3 - 2$

$\therefore x \in \{-3, -2, 7\}$

$\therefore n = 3$

Hence, $m^2 + mx + n^2 = 0 + 0 + 9 = 9$

HINT:

The relation between the greatest integer function and fractional part is :

$[x] = x - \{x\}$

23. Correct answer is [11].

Equation of plane P_2 passing through (2, –1, 0), (2, 0, –1) and (5, 1, 1) is

$$\begin{vmatrix} x-5 & y-1 & z-1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow (x-5)(4-1) - (y-1)(6-3) + (z-1)(3-6) = 0$

$\Rightarrow 3x - 15 - 3y + 3 - 3z + 3 = 0$

$\Rightarrow 3x - 3y - 3z - 9 = 0$

$\Rightarrow x - y - z = 3 \quad$ ...(i)

Now, direction ratios of line of intersection of P_1 and P_2 is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 3 & -1 & -7 \end{vmatrix}$$

$= \hat{i}(7-1) - \hat{j}(-7+3) + \hat{k}(-1+3)$

$= 6\hat{i} + 4\hat{j} + 2\hat{k}$

At $z = 0$, $x - y = 3$ [from (i)]

$3x - y = 11$

on solving, we get

$x = 4$ and $y = 1$

So, equation of line is

$$\frac{x-4}{6} = \frac{y-1}{4} = \frac{z-2}{6} = k$$

$\therefore (\alpha, \beta, \gamma) = (6k + 4, 4k + 1, 2k)$

$\Rightarrow (6)(\alpha - 7) + 4(\beta - 4) + 2(\gamma + 1) = 0$

$\Rightarrow 6(6k + 4 - 7) + 4(4k + 1 - 4) + 2(2k + 1) = 0$

$\Rightarrow 36k - 18 + 16k - 12 + 4k + 4 = 0$

$\Rightarrow 56k = 26 \Rightarrow k = \frac{1}{2}$

So, $\alpha = 7$, $\beta = 3$ and $\gamma = 1$

$\therefore \alpha + \beta + \gamma = 7 + 3 + 1 = 11$

24. Correct answer is [20].

Domain of $\log_e\left(\frac{6x^2+5x+1}{2x-1}\right)$

So, $\frac{6x^2+5x+1}{2x-1} > 0$

$\Rightarrow \frac{(3x+1)(2x+1)}{2x-1} > 0$

$\Rightarrow x \in \left(\frac{-1}{2}, \frac{-1}{3}\right) \cup \left(\frac{1}{2}, \infty\right)$

For domain of $\cos^{-1}\left(\frac{2x^2-3x+4}{3x-5}\right)$ domain of $\cos^{-1}x \to [-1, 1]$

$$-1 \le \frac{2x^2-3x+4}{3x-5} \le 1$$

$\frac{2x^2-1}{3x-5} \ge 0$ and $\frac{2x^2-6x+9}{3x-5} \le 0$

$\Rightarrow x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \cup \left(\frac{5}{3}, \infty\right)$

So, common domain is $\left(\frac{-1}{2}, \frac{-1}{3}\right) \cup \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$

$\therefore 18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 18\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{2}\right)$

$= 18\left(\frac{9+4+9+18}{36}\right) = \frac{1}{2}(40) = 20$

HINT:

For $\log_e x$, $x > 0$ and $-1 \le \cos^{-1} x \le 1$

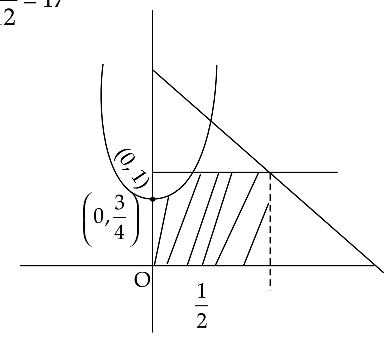
25. Correct answer is [17].

$$\text{Required area} = \left[\int_0^{\frac{1}{2}} \left(x^2 + \frac{3}{4}\right) dx\right] + \left[\frac{1}{2}\left(\frac{3}{2} + \frac{1}{2}\right) \times 1\right]$$

$$= \left[\frac{x^3}{3} + \frac{3x}{4}\right]_0^{\frac{1}{2}} + 1$$

$$= \frac{1}{24} + \frac{3}{8} - 0 + 1 = \frac{1+9+24}{24} = \frac{34}{24} = \frac{17}{12}$$

So, $12A = 12 \times \frac{17}{12} = 17$



HINT:

Find the common region bounded by all the given curves and then using integration, find the required area.

26. Correct answer is [150].

$\because \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\Rightarrow \frac{1}{x} + \frac{1}{z} = \frac{2}{y}$ ...(i)

and $x, \sqrt{2}y, z$ are in G.P.

$\Rightarrow 2y^2 = xz$ ...(ii)

from (i), $\frac{2}{y} = \frac{x+z}{xz} = \frac{x+z}{2y^2}$

$\Rightarrow 4y = x + z$

Also, $xy + yz + zx = \frac{3}{\sqrt{2}}xyz$

$y(4y) + xz = \frac{3}{\sqrt{2}}(2y^2)y$

$\Rightarrow 4y^2 + 2y^2 = 3\sqrt{2}y^3$

$\Rightarrow 6y^2 = 3\sqrt{2}y^3 \Rightarrow y = \sqrt{2}$

$\therefore 3(x+y+z)^2 = 3(5y)^2 = 3(5\sqrt{2})^2$

$= 150$

27. Correct answer is [12].

Given:

$(\cos y), (\ln(\cos y))^2\, dx = (1 + 3x \ln \cos y) \sin y\, dy$

$\Rightarrow \frac{dx}{dy} = \frac{(1 + 3x \ln \cos y)\sin y}{(\ln \cos y)^2 \cos y}$

$= \tan y\left[\frac{1}{(\ln \cos y)^2} + \frac{3x}{\ln \cos y}\right]$

$\Rightarrow \frac{dx}{dy} - \left(\frac{3\tan y}{\ln \cos y}\right)x = \frac{\tan y}{(\ln \cos y)^2}$

which is a linear differential equation.

$$\text{I.F.} = e^{-\int \frac{3\tan y}{\ln \cos y} dy} = (\ln \cos y)^3 \qquad \text{I.F.} = e^{\int P.dx}$$

So, the solution is :

$$x \times (\ln \cos y)^3 = \int \left((\ln \cos y)^3 \times \frac{\tan y}{(\ln \cos y)^2} \right) dy$$

$$x \times (\ln \cos y)^3 = \frac{-(\ln \cos y)^2}{2} + C$$

At $y = \frac{\pi}{3}$,

$$\frac{1}{2\ln 2} \times \left(\ln\left(\frac{1}{2}\right)\right)^3 = -\frac{\left(\ln\left(\frac{1}{2}\right)\right)^2}{2} + C$$

$\Rightarrow C = 0$

So, $x \times \ln^3 \cos y = \dfrac{-\ln^2 \cos y}{2}$

At $y = \frac{\pi}{6}, x \times \left(\ln\left(\frac{\sqrt{3}}{2}\right)\right)^3 = -\frac{1}{2}\left(\ln\left(\frac{\sqrt{3}}{2}\right)\right)^2$

$$\Rightarrow x = -\frac{1}{2\ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$= -\frac{1}{2[\ln\sqrt{3} - \ln 2]} = \frac{-1}{2\left[\frac{1}{2}\ln 3 - \ln 2\right]}$$

$$= \frac{-1}{2\left[\frac{\ln 3 - \ln 4}{2}\right]} = \frac{1}{\ln 4 - \ln 3}$$

$\Rightarrow m = 4, n = 3$

$\Rightarrow mn = 12$

28. Correct answer is [6].

$$\int_0^{2.4} [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^{2} [x^2]dx + \int_{2}^{\sqrt{5}} [x^2]dx + \int_{\sqrt{5}}^{2.4} [x^2]dx$$

$$= \int_0 0.dx + \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3dx + \int_{2}^{\sqrt{5}} 4dx + \int_{\sqrt{5}}^{2.4} 5dx$$

$$0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{\sqrt{3}} + 3[x]_{\sqrt{3}}^{2} + 4[x]_2^{\sqrt{5}} + 5[x]_{\sqrt{5}}^{2.4}$$

$$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} + 4\sqrt{5} - 8 + 12 - 5\sqrt{5}$$

$$= -\sqrt{2} - \sqrt{3} - \sqrt{5} + 9$$

$\therefore \alpha = 9, \beta = -1, \gamma = -1, \delta = -1$

So, $\alpha + \beta + \gamma + \delta = 9 - 1 - 1 - 1 = 6$

HINT:

The greater integer value is that integeral value which is less than or equal to that number.

29. Correct answer is [16].

Give parabola is : $y^2 = 12x$ $\qquad (\because a = 3)$

So, $P \equiv (at_1^2, 2at_1)$

$Q \equiv (at_2^2, 2at_2)$

So, point R $(\alpha, \beta) \equiv (at_1t_2, a(t_1 + t_2))$

$\equiv ((3t)(3t), 3(t + 3t)) = (9t^2, 12t)$

$$\therefore \frac{\beta^2}{\alpha} = \frac{144t^2}{9t^2} = 16$$

HINT:

For equation of parabola $y^2 = 4ax$, focus is $(a, 0)$

30. Correct answer is [309].

$$\text{Here, } f(x) = \begin{cases} 3x^2 + k\sqrt{x+1}, & 0 < x < 1 \\ mx^2 + k^2, & x \geq 1 \end{cases}$$

$\because f(x)$ is differentiable at $x > 0$

So, $f(x)$ is differentiable at $x = 1$

$f(1^-) = f(1) = f(1^+)$

$$3 + k\sqrt{2} = m + k^2 \qquad \text{...(i)}$$

$f'(1^-) = f'(1^+)$

$$6(1) + \frac{k}{2\sqrt{1+1}} = 2m(1)$$

$$\Rightarrow 6 + \frac{k}{2\sqrt{2}} = 2m \qquad \text{...(ii)}$$

Using (i) and (ii),

$$3 + k\sqrt{2} = 3 + \frac{k}{4\sqrt{2}} + k^2$$

$$\Rightarrow k^2 + k\left[\frac{1}{4\sqrt{2}} - \sqrt{2}\right] = 0$$

$$\Rightarrow k\left[k + \frac{1-8}{4\sqrt{2}}\right] = 0 \Rightarrow k = 0, \frac{7}{4\sqrt{2}}$$

$$\text{for } k = \frac{7}{4\sqrt{2}}, m = 3 + \frac{\frac{7}{4\sqrt{2}}}{4\sqrt{2}}$$

$$= 3 + \frac{7}{32} = \frac{96+7}{32} = \frac{103}{32}$$

$$\text{So, } \frac{8f'(8)}{f'\left(\frac{1}{8}\right)} = \frac{8 \times \left[2 \times \frac{103}{32} \times 8\right]}{6 \times \frac{1}{8} + \frac{7}{4\sqrt{2}} \times 2\sqrt{918}}$$

$$= \frac{412}{\frac{3}{4} + \frac{7}{12}} = \frac{412}{\frac{9+7}{12}} = \frac{412 \times 12}{16} = 309$$

HINT:

$f(x)$ is differentiable at $x = a$, if $f'(a^-) = f'(a^+)$

❑❑