ISC Solved Paper 2019

Class-XII

Mathematics

(Maximum Marks : 80)

(Time allowed : Three hours)

*Condidates are allowed an additional **15 minutes** for **only** reading the paper.*

*They must **NOT** start writing during this time.*

The question Paper consists of three sections A, B and C.

*Candidates are required to attempt all questions from **Section A** and all questions*

EITHER** from **Section B** **OR** **Section C

***Section A :** Internal choice has been provided in three questions of four marks each and two question of six marks each.*

***Section B :** Internal choice has been provided in two questions of four marks each.*

***Section C :** Internal choice has been provided in two questions of four marks each.*

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

SECTION - A **[80 Marks]**

1. **[10×2]**

***(i)** **If $f : R\to R, f(x) = x^3$ and $g : R\to R$, $g(x) = 2x^2+1$, and R is the set of real numbers, then find $fog\,(x)$ and $gof\,(x)$.**

(ii) **Solve: $\sin(2\tan^{-1}x) = 1$**

Ans. $\sin(2\tan^{-1}x) = 1$

$\Rightarrow \quad 2\tan^{-1}x = \sin^{-1}1$

$$2\tan^{-1}x = \frac{\pi}{2}$$

or, $\tan^{-1}x = \dfrac{\pi}{4}$

or, $x = \tan\dfrac{\pi}{4} = 1$

(iii) **Using determinants, find the values of k, if the area of triangle with vertices (–2, 0), (0, 4) and (0, k) is 4 square units.**

Ans. Area of triangle $= \dfrac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$

$$\frac{1}{2}\left[-2(4-k)\right] = 4 \Rightarrow k-4 = 4$$

$\therefore k = 8$

(iv) **Show that $(A + A')$ is symmetric matrix, if $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$.**

Ans. Given, $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Now, $A + A^T$ (say)

$$= \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} = B$$

$$B^T = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix}$$

$\because B^T = B \Rightarrow B$ is a Symmetric Matrix

(v) **$f(x) = \dfrac{x^2-9}{x-3}$ is not defined at $x = 3$. What value should be assigned to $f(3)$ for continuity of $f(x)$ at $x = 3$?**

Ans.

$$\text{LHL} \lim_{h\to 0} f(3-h) = \lim_{h\to 0}\frac{(3-h)^2-9}{(3-h)-3} = \lim_{h\to 0}\frac{9+h^2-6h-9}{-h}$$

$$\lim_{h\to 0}\frac{h(h-6)}{-h} = \lim_{h\to 0} 6-h = 6$$

$$\text{RHL} \lim_{h\to 0} f(3+h) = \lim_{h\to 0}\frac{(3+h)^2-9}{(3+h)-3} = \lim_{h\to 0}\frac{9+h^2+6h-9}{h}$$

$$\lim_{h\to 0}\frac{h(h+6)}{h} = \lim_{h\to 0} h+6 = 6$$

Value of function assigned should be $f(3) = 6$

(vi) **Prove that the function $f(x) = x^3 - 6x^2 + 12x + 5$ is increasing on R.**

Ans. Since, $f(x) = x^3 - 6x^2 + 12x + 5$,

***Out of Syllabus**

$f'(x) = 3x^2 - 12x + 12$
$= 3[x^2 - 4x + 4]$
$= 3[x-2]^2$

$(x-2)^2$ is a $+ve$ quantity and further thrice of it will always be $+ve$.

Hence $f'(x) > 0 \ \forall \ x$

$\therefore$ Increasing $f(x)$ is increasing on R.

(vii) Evaluate: $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx$

Ans.

$$\int \frac{\sec^2 x}{\text{cosec}^2 x} dx$$

$$\int \frac{\sin^2 x}{\cos^2 x} dx = \int \tan^2 x \, dx = \int (\sec^2 x - 1) dx$$

$$= \tan x - x + C$$

(viii) Using L' Hospital's Rule, evaluate: $\lim\limits_{x \to 0} \frac{8^x - 4^x}{4x}$

Ans. $\lim\limits_{x \to 0} \frac{8^x - 4^x}{4x} \quad \left(\frac{0}{0}\right)$ form

Applying L' Hospital's rule;

$$\lim_{x \to 0} \frac{8^x \log 8 - 4^x \log 4}{4}$$

$$= \frac{8^0 \log 8 - 4^0 \log 4}{4}$$

$$= \frac{\log 8 - \log 4}{4} = \frac{1}{4} \log 2$$

(ix) Two balls are drawn from an urn containing 3 white, 5 red and 2 black balls, one by one without replacement. What is the probability that at least one ball is red?

Ans. Types of balls, $W = 3$, $R = 5$ and $B = 2$

P(at least one red) $= 1 - P$(none Red)

$$= 1 - \left(\frac{5}{10} \times \frac{4}{9}\right) = 1 - \frac{20}{90}$$

$$= \frac{7}{9}$$

(x) If events A and B are independent, such that P(A) $= \frac{3}{5}$, $P(B) = \frac{2}{3}$, find $P(A \cup B)$.

Ans. Given : $P(A) = \frac{3}{5}, P(B) = \frac{2}{3}$

$$\because P(A \cup B) = P(A) + P(B) - P(A).P(B)$$

$$= \frac{3}{5} + \frac{2}{3} - \frac{3}{5} \times \frac{2}{3}$$

$$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$$

$$= \frac{9 + 10 - 6}{15}$$

$$= \frac{13}{15}$$

2. If $f : A \to A$ and $A = R - \left\{\frac{8}{5}\right\}$, show that the function $f(x) = \frac{8x+3}{5x-8}$ is one – one onto. Hence, find f^{-1}. [4]

Ans. $f(x) = \frac{8x+3}{5x-8}$

$$f(x_1) = \frac{8x_1 + 3}{5x_1 - 8} \quad \text{and} \quad f(x_2) = \frac{8x_2 + 3}{5x_2 - 8}$$

Put $f(x_1) = f(x_2)$

$$\frac{8x_1 + 3}{5x_1 - 8} = \frac{8x_2 + 3}{5x_2 - 8}$$

$$(8x_1 + 3)(5x_2 - 8) = (8x_2 + 3)(5x_1 - 8)$$

$$40x_1x_2 - 64x_1 + 15x_2 - 24 = 40x_1x_2 - 64x_2 + 15x_1 - 24$$

$$79x_2 = 79x_1$$

$$\therefore x_1 = x_2$$

Hence, $f(x)$ one-one

$$y = \frac{8x+3}{5x-8}$$

$$5xy - 8y = 8x + 3$$

$$\Rightarrow \quad 5xy - 8x = 3 + 8y$$

$$x = \frac{3 + 8y}{5y - 8}$$

x is undefined for $y = \frac{8}{5}$ which is already excluded from Range.

Hence function is onto

$$\Rightarrow f^{-1}(y) = \frac{3 + 8y}{5y - 8}$$

or $f^{-1}(x) = \frac{3 + 8x}{5x - 8}$

3.(a) Solve for x : $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$ [4]

OR

(b) If $\sec^{-1} x = \text{cosec}^{-1} y$, show that $\frac{1}{x^2} + \frac{1}{y^2} = 1$

Ans. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\therefore \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2} \times \frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\text{or,} \left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2} \times \frac{x+1}{x+2}\right)}\right] = \tan\frac{\pi}{4}$$

$$\text{or,} \frac{(x-1)(x+2) + (x+1)(x-2)}{(x+2)(x-2) - (x-1)(x+1)} = 1$$

$$\text{or,} \frac{(x^2 + x - 2) + (x^2 - x - 2)}{(x^2 - 4) - (x^2 - 1)} = 1$$

or, $\frac{2x^2-4}{-3}=1 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}$.

OR

(b) Since, $\sec^{-1}x=\operatorname{cosec}^{-1}y$

$$\cos^{-1}\frac{1}{x}=\sin^{-1}\frac{1}{y}$$

$$\cos^{-1}\frac{1}{x}=\cos^{-1}\frac{\sqrt{y^2-1}}{y}$$

$$\frac{1}{x}=\frac{\sqrt{y^2-1}}{y}$$

$$\frac{1}{x^2}=\frac{y^2-1}{y}$$

$$\frac{1}{x^2}=1-\frac{1}{y^2}$$

$$\frac{1}{x^2}+\frac{1}{y^2}=1$$

4. Using properties of determinants prove that:

$$\begin{vmatrix} x & x(x^2+1) & x+1 \\ y & y(y^2+1) & y+1 \\ z & z(z^2+1) & z+1 \end{vmatrix}=(x-y)(y-z)(z-x)(x+y+z)$$

[4]

Ans. $\begin{vmatrix} x & x(x^2+1) & x+1 \\ y & y(y^2+1) & y+1 \\ z & z(z^2+1) & z+1 \end{vmatrix}$

$$=\underbrace{\begin{vmatrix} x & x(x^2+1) & x \\ y & y(y^2+1) & y \\ z & z(z^2+1) & z \end{vmatrix}}_{\text{Zero as } C_1=C_3}+\begin{vmatrix} x & x(x^2+1) & 1 \\ y & y(y^2+1) & 1 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

$$=\begin{vmatrix} x & x(x^2+1) & 1 \\ y & y(y^2+1) & 1 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

Apply, $R_1 \to R_1-R_2;\ R_2 \to R_2-R_3$

$$=\begin{vmatrix} x-y & x(x^2+1)-y(y^2+1) & 0 \\ y-z & y(y^2+1)-z(z^2+1) & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

$$=\begin{vmatrix} x-y & (x-y)(x^2+y^2+xy)+(x-y) & 0 \\ y-z & (y-z)(y^2+z^2+yz)+(y-z) & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} 1 & x^2+y^2+xy+1 & 0 \\ 1 & y^2+z^2+yz+1 & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

Apply, $R_1 \to R_1-R_2$

$$=(x-y)(y-z)\begin{vmatrix} 0 & x^2-z^2+xy-yz & 0 \\ 1 & y^2+z^2+yz+1 & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

$$=(x-y)(y-z)(x-z)\begin{vmatrix} 0 & x+z+y & 0 \\ 1 & y^2+z^2+yz+1 & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

$$=(x-y)(y-z)(x-z)(x+y+z)\begin{vmatrix} 0 & 1 & 0 \\ 1 & y^2+z^2+yz+1 & 0 \\ z & z(z^2+1) & 1 \end{vmatrix}$$

Expanding the determinant;

$$=(x-y)(y-z)(x-z)(x+y+z)[-1]$$

$$=(x-y)(y-z)(z-x)(x+y+z)$$

5.(a) Show that the function $f(x)=|x-4|,\ x\in R$ is continuous, but not differentiable at $x=4$. [4]

OR

(b) Verify the Lagrange's mean value theorem for the function : $f(x)=x+\frac{1}{x}$ in the interval [1, 3]

Ans. (a) $f(x)=|x-4|$ and $a=4$

$$\text{LHL}=\lim_{h\to 0}f(4-h)=\lim_{h\to 0}|(4-h)-4|=\lim_{h\to 0}h=0$$

$$\text{RHL}=\lim_{h\to 0}f(4+h)=\lim_{h\to 0}|(4+h)-4|=\lim_{h\to 0}h=0$$

Value of function; $f(4)=|4-4|=0$

Hence $f(x)$ is continuous

$$\text{LHD}=\lim_{h\to 0}\frac{f(4-h)-f(4)}{-h}=\lim_{h\to 0}\frac{h-0}{-h}=-1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1$$

Hence $f(x)$ is not differentiable at $x=4$

OR

(b) Given : function $f(x)=x+\frac{1}{x}$

$f(x)$ is continuous in [1, 3]

$f(x)$ is differentiable in (1, 3)

$$f'(x)=1-\frac{1}{x^2}$$

$$f'(c)=1-\frac{1}{c^2}$$

$$f(a)=f(1)=2$$

$$f(b)=f(3)=\frac{10}{3}$$

$$\because f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$1-\frac{1}{c^2}=\frac{\frac{10}{3}-2}{3-1}$$

$$\frac{1}{c^2}=1-\frac{2}{3}$$

$$\frac{1}{c^2}=\frac{1}{3}\Rightarrow c^2=3\Rightarrow c=\sqrt{3}$$

$$c=1.732$$

$$c=1.732\in(1,3)$$

$\therefore$ Lagrange theorem is verified

6. If $y=e^{\sin^{-1}x}$ and $z=e^{-\cos^{-1}x}$, prove that $\frac{dy}{dz}=e^{\frac{\pi}{2}}$ **[4]**

Ans. Given: $y=e^{\sin^{-1}x}$ and $z=e^{-\cos^{-1}x}$

$$\frac{y}{z}=\frac{e^{\sin^{-1}x}}{e^{-\cos^{-1}x}}$$

$$\frac{y}{z}=e^{\sin^{-1}x+\cos^{-1}x} \quad \left(\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$$

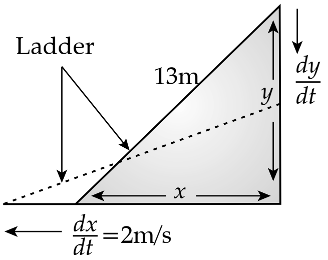
$$\frac{y}{z}=e^{\frac{\pi}{2}}\Rightarrow y=e^{\frac{\pi}{2}}z$$

Differentiating above w.r.t z,

$$\frac{dy}{dz}=e^{\frac{\pi}{2}}$$

7. A 13 m long ladder is leaning against a wall, touching the wall at a certain height from the ground level. The bottom of the ladder is pulled away from the wall, along the ground, at the rate of 2 m/s. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? **[4]**

Ans.



From the figure,

$$x^2+y^2=13^2$$

or, $x^2+y^2=169$

Differentiating above

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=\frac{-x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-5}{12}(2)$$

$$\frac{dy}{dt}=\frac{-5}{6}\text{ms}^{-1}$$

8. (a) Evaluate: $\int\frac{x(1+x^2)}{1+x^4}dx$ **[4]**

OR

(b) Evaluate: $\int_{-6}^{3}|x+3|dx$

Ans. (a) $\int\frac{x(1+x^2)}{1+x^4}dx$

$$\int\frac{x}{1+x^4}dx+\int\frac{x^3}{1+x^4}dx$$

put $x^2=t$, $2x=\frac{dt}{dx}$ | put $x^4=t$, $4x^3=\frac{dt}{dx}$

$$=\frac{1}{2}\int\frac{dt}{1+t^2}+\frac{1}{4}\int\frac{dt}{1+t}$$

$$=\frac{1}{2}\tan^{-1}t+\frac{1}{4}\log(1+t)$$

$$=\frac{1}{2}\tan^{-1}(x^2)+\frac{1}{4}\log(1+x^4)+C$$

OR

(b) $\int_{-6}^{3}|x+3|dx$ $\quad\because x+3=0$, $x=-3$

$$=\int_{-6}^{-3}-(x+3)dx+\int_{-3}^{3}(x+3)dx$$

$$=-\left[\frac{x^2}{2}+3x\right]_{-6}^{-3}+\left[\frac{x^2}{2}+3x\right]_{-3}^{3}$$

$$=-\left[\frac{1}{2}\{(-3)^2-(-6)^2\}+3\{(-3)-(-6)\}\right]+\left[\frac{1}{2}\{(3)^2-(-3)^2\}+3\{(3)-(-3)\}\right]$$

$$=-\left[\frac{1}{2}\{9-36\}+3\{-3+6\}\right]+\left[\frac{1}{2}\{9-9\}+3\{3+3\}\right]$$

$$=-\left[\frac{1}{2}\{-27\}+9\right]+\left[\frac{1}{2}\{0\}+3\{6\}\right]$$

$$=-[-13.5+9]+[18]$$

$$=4.5+18=22.5$$

9. Solve the differential equation: $\frac{dy}{dx}=\frac{x+y+2}{2(x+y)-1}$ **[4]**

Ans. $\frac{dy}{dx}=\frac{(x+y)+2}{2(x+y)-1}$ … (i)

Let, $x+y=v$

$1+\frac{dy}{dx}=\frac{dv}{dx}$ …(ii)

From equations (i) and (ii),

$$\frac{dv}{dx}-1=\frac{v+2}{2v-1}$$

or, $\frac{dv}{dx}=\frac{v+2}{2v-1}+1$

or, $\frac{dv}{dx}=\frac{v+2+2v-1}{2v-1}$

or, $\frac{dv}{dx}=\frac{3v+1}{2v-1}$

or, $\frac{2v-1}{3v+1}dv=dx$

Integrating above differential equation

$$2\int \frac{v-\frac{1}{2}}{3v+1}dv = \int dx$$

or, $\frac{2}{3}\int \frac{3v-\frac{3}{2}}{3v+1}dv = x$

or, $\frac{2}{3}\int \frac{3v+1-\frac{5}{2}}{3v+1}dv = x$

or, $\frac{2}{3}\int 1dv - \frac{5}{3}\int \frac{dv}{3v+1} = x$

or, $\frac{2}{3}v - \frac{5}{3}\frac{\log(3v+1)}{3} = x$

or, $\frac{2}{3}v - \frac{5}{9}\log(3v+1) = x + C$

Put $v = x + y$

$$\frac{2}{3}(x+y) - \frac{5}{9}\log[3(x+y)+1] = x + C$$

10. Bag A contains 4 white balls and 3 black-balls, while Bag B contains 3 white balls and 5 black balls. Two balls are drawn from Bag A and placed in Bag B. Then, what is the probability of drawing a white ball from Bag B? [4]

Ans. Bag A has $4W$ and $3B$ balls. Bag B has $3W$ and $5B$ balls.

$$P\left(\frac{W}{\text{Bag}\,B}\right) = P\left(\frac{BB}{\text{Bag}\,A}\right)\bullet P\left(\frac{W}{\text{Bag}\,B}\right)$$

or $P\left(\frac{WW}{\text{Bag}\,A}\right)\bullet P\left(\frac{W}{\text{Bag}\,B}\right)$

or $P\left(\frac{WB}{\text{Bag}\,A}\right)\bullet P\left(\frac{W}{\text{Bag}\,B}\right)$

$$= \frac{^3C_2}{^7C_2}\times\frac{3}{10} + \frac{^4C_2}{^7C_2}\times\frac{5}{10} + \frac{^4C_1\times{}^3C_1}{^7C_2}\times\frac{4}{10}$$

$$= \frac{3}{21}\times\frac{3}{10} + \frac{6}{21}\times\frac{5}{10} + \frac{4\times3}{21}\times\frac{4}{10}$$

$$= \frac{9}{210} + \frac{30}{210} + \frac{48}{210} = \frac{87}{210} = \frac{29}{70}$$

11. Solve the following system of linear equations using matrix method:

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9;\ \frac{2}{x}+\frac{5}{y}+\frac{7}{z}=52;\ \frac{2}{x}+\frac{1}{y}-\frac{1}{z}=0$ [6]

Ans. Let, $\frac{1}{x}=a,\ \frac{1}{y}=b,\ \frac{1}{z}=c,$ So the given equations can be written as

$$a+b+c=9,\quad 2a+5b+7c=52,\quad 2a+b-c=0$$

Let, $A = \begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{bmatrix}, B = \begin{bmatrix}9\\ 52\\ 0\end{bmatrix}$

$$|A| = 1(-12) - 1(-16) + 1(-8)$$

$$|A| = -12 + 16 - 8$$

$$|A| = -4 = (-\text{ve})$$

Cofactors of A are

$$a_{11} = \begin{vmatrix}5 & 7\\ 1 & -1\end{vmatrix} = (-5-7) = -12$$

$$a_{12} = -\begin{vmatrix}2 & 7\\ 2 & -1\end{vmatrix} = -(-2-14) = 16$$

$$a_{13} = \begin{vmatrix}2 & 5\\ 2 & 1\end{vmatrix} = (2-10) = -8$$

$$a_{21} = -\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix} = -(-1-1) = 2$$

$$a_{22} = \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix} = (-1-2) = -3$$

$$a_{23} = -\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix} = -(1-2) = 1$$

$$a_{31} = \begin{vmatrix}1 & 1\\ 5 & 7\end{vmatrix} = (7-5) = 2$$

$$a_{32} = -\begin{vmatrix}1 & 1\\ 2 & 7\end{vmatrix} = -(7-2) = -5$$

$$a_{33} = \begin{vmatrix}1 & 1\\ 2 & 5\end{vmatrix} = (5-2) = 3$$

$$adjoint\ A = \begin{bmatrix}-12 & 16 & -8\\ 2 & -3 & 1\\ 2 & -5 & 3\end{bmatrix}^T$$

$$\text{Adjoint } A = \begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}.\text{adj }(A) = \frac{1}{-4}\begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix}$$

$$X = A^{-1}.B$$

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \frac{1}{-4}\begin{bmatrix}-12 & 2 & 2\\ 16 & -3 & -5\\ -8 & 1 & 3\end{bmatrix}\begin{bmatrix}9\\ 52\\ 0\end{bmatrix}$$

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \frac{1}{-4}\begin{bmatrix}-108+104+0\\ 144-156+0\\ -72+52+0\end{bmatrix}$$

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \frac{-1}{4}\begin{bmatrix}-4\\ -12\\ -20\end{bmatrix}$$

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}1\\ 3\\ 5\end{bmatrix} \Rightarrow x = 1,\ y = \frac{1}{3},\ z = \frac{1}{5}$$

12. (a) The volume of a closed rectangular metal box with a square base is 4096 cm^3. The cost of polishing the outer surface of the box is ₹ 4 per cm^2. Find the dimensions of the box for the minimum cost of polishing it.

OR

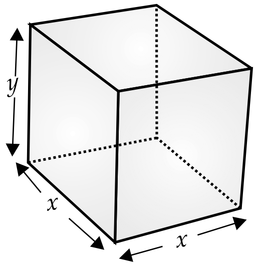
(b) Find the point on the straight line $2x + 3y = 6$, which is closest to the origin. [6]

Ans. Let x be the dimension of base and y be the height of the rectangular box

According to the problem

$x^2y = 4096$ … (i)

Total surface area = $(4xy + 2x^2)$



Total cost, $C = \left[4xy + 2x^2\right]4$

$$C = 4\left[4x \cdot \frac{4096}{x^2} + 2x^2\right]$$

$$C = 4\left[\frac{4\times 4096}{x} + 2x^2\right]$$

Differentiating above,

$$\frac{dC}{dx} = 4\left[\frac{-4\times 4096}{x^2} + 4x\right]$$

For maxima and minima

$$\frac{dC}{dx} = 0 \Rightarrow 4x = \frac{4\times 4096}{x^2}$$

$$x^3 = 4096$$

$$x = \sqrt[3]{4096}$$

$$x = 16$$

Now, $\dfrac{d^2C}{dx^2} = 4\left[\dfrac{8\times 4096}{x^3} + 4\right]$

$$\left(\frac{d^2C}{dx^2}\right)_{x=16} = +\text{ve (Minimum)}$$

Putting $x = 16$ in eq.(1) we get $y = \dfrac{4096}{16\times 16} \Rightarrow y = 16$

Hence the required dimensions of the box will be of size 16cm for its minimum cost.

OR

(b) Let the point be P(x, y) on the line $2x+3y=6$ ....(1)

Minimum distance $d = \sqrt{x^2 + y^2}$

$d^2 = F = x^2 + y^2$ [Here, $d^2 = F$]

$$F = x^2 + \left(\frac{6-2x}{3}\right)^2$$

...using equation(1)

$$\frac{dF}{dx} = 2x + 2\left(\frac{6-2x}{3}\right)\left(-\frac{2}{3}\right)$$

For maxima and minima

$$\frac{dF}{dx} = 0 \Rightarrow x = \frac{2(6-2x)}{9}$$

or, $9x = 12 - 4x$

or, $13x = 12 \Rightarrow x = \dfrac{12}{13}$

$$\frac{d^2F}{dx^2} = 2 - \frac{4}{9}[-2] = 2 + \frac{8}{9} = +\text{ve(Minima)}$$

Putting the value of x in eq(1);

$$3y = 6 - 2\left(\frac{12}{13}\right)$$

or, $3y = 6 - \dfrac{24}{13}$

or, $y = \dfrac{18}{13}$

Hence, the required Point $P\left(\dfrac{12}{13}, \dfrac{18}{13}\right)$

13. Evaluate: $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ [6]

Ans. $I = \displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ …(1)

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}dx$$

$$I = \int_0^{\pi} \frac{(x-\pi)\tan x}{-(\sec + \tan x)}dx$$

$I = \displaystyle\int_0^{\pi} \frac{(\pi - x)\tan x}{(\sec + \tan x)}dx$ …(2)

Adding equation (1) and (2),

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x}dx$$

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x}dx$$

$$2I = \pi\int_0^{\pi}\left(\frac{\tan x}{\sec x + \tan x} \times \frac{\sec x - \tan x}{\sec x - \tan x}\right)dx$$

$$2I = \pi\int_0^{\pi} \tan x\,(\sec x - \tan x)\,dx$$

$$2I = \pi\left[\int_0^{\pi} \sec x\tan x\,dx - \int_0^{\pi}\tan^2 x\,dx\right]$$

$$2I = \pi\left[(\sec x)_0^{\pi} - (\tan x - x)_0^{\pi}\right]$$

$$2I = \pi\left[(\sec\pi - \sec 0) - (\tan\pi - \tan 0) - (\pi - 0)\right]$$

$$2I = \pi\left[\{(-1) - 1\} - \{0 - 0\} - \pi\right]$$

$$2I = \pi[-2+\pi]$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

14. (a) Given three identical Boxes *A, B* and *C*, Box *A* contains 2 gold and 1 silver coin, Box *B* contains 1 gold and 2 silver coins and Box *C* contains 3 silver coins. A person chooses a Box at random and takes out a coin. If the coin drawn is of silver, find the probability that it has been drawn from the Box which has the remaining two coins also of silver.

OR

(b) Determine the binomial distribution where mean is 9 and standard deviation is $\frac{3}{2}$. Also, find the probability of obtaining at most one success. [6]

Ans. (a) Box A has 2 *G* and 1 *S* coins

Box A has 1 *G* and 2 *S* coins

Box C has 0 *G* and 3 *S* coins

The required probability $P\left(\frac{\text{Box C}}{\text{S}}\right)$ can be obtained by using Bayes' Theorem, which is given as

$$= \frac{P\left(\frac{S}{Box\,C}\right)\cdot P(Box\,C)}{P\left(\frac{S}{Box\,A}\right)\cdot P(Box\,A) + P\left(\frac{S}{Box\,B}\right)\cdot P(Box\,B) + P\left(\frac{S}{Box\,C}\right)\cdot P(Box\,C)}$$

$$= \frac{\frac{3}{3}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + \frac{3}{3}\cdot\frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{9}+\frac{2}{9}+\frac{3}{9}}$$

$$= \frac{\frac{1}{3}}{\frac{6}{9}}$$

$$= \frac{1}{2}$$

OR

(b) Here, $np = 9$ ...(i)

So, $npq = \frac{9}{4}$

$$9q = \frac{9}{4} \Rightarrow q = \frac{1}{4}$$

$$p = \frac{3}{4}$$

From equation (i),

$$n\left(\frac{3}{4}\right) = 9$$

$$n = 12$$

P(at most success)

$$= {}^{12}C_0\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{12} + {}^{12}C_1\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{11}$$

$$= \left(\frac{1}{4}\right)^{12} + 12\times\frac{3}{4}\times\left(\frac{1}{4}\right)^{11}$$

$$= \left(\frac{1}{4}\right)^{11}\left[\frac{1}{4}+9\right]$$

$$= \left(\frac{1}{4}\right)^{11}\left[\frac{37}{4}\right] = 37\left(\frac{1}{4}\right)^{12}$$

SECTION - B [20 Marks]

15. (a) If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|$ = 13 and $|\vec{a}|$ = 5, find the value of $|\vec{b}|$. [3×2]

(b) Find the length of the perpendicular from origin to the plane $\vec{r}.(3i-4j-12k)+39=0$.

(c) Find the angle between the two lines $2x = 3y = -z$ and $6x = -y = -4z$.

Ans. (a) *a* and *b* are perpendicular.

so, $\vec{a}.\vec{b} = 0$

Given : $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$

So, $|\vec{a}+\vec{b}|^2 = 169$

$$(\vec{a}+\vec{b})(\vec{a}+\vec{b}) = 169$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 169$$

$$25 + |\vec{b}|^2 + 2.0 = 169$$

$$|\vec{b}|^2 = 144$$

(b) The equation can be written as,

$$3x - 4y - 12z + 39 = 0$$

Distance from origin $= \frac{39}{\sqrt{3^2+(-4)^2+(-12)^2}}$

$= \frac{39}{13}$ = 3 units

(c) Equation of first line in cartesian form

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-6}$$

Direction Ratios are (3, 2, –6)

Equation of second line in cartesian form

$$\frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

Direction Ratios are (2, –12, –3)

$$\cos\theta = \frac{6-24+18}{\sqrt{9+4+36}\sqrt{4+144+9}}$$

$$\cos\theta = \frac{0}{7\sqrt{157}} = 0$$

$\theta = 90°$

16. (a) If $\vec{a} = i - 2j + 3k$, $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, prove that $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular.

OR

(b) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of x such that the vectors $\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.

Ans. **(a)** Given: $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and

$\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$\vec{a}\times\vec{b} = \hat{i} + 11\hat{j} + 7\hat{k}$

$\vec{a}.\left(\vec{a}\times\vec{b}\right) = (i - 2j + 3k)(i + 11j + 7k)$

$= 1 - 22 + 21 = 0$

Hence $\vec{a} \perp (\vec{a}\times\vec{b})$

OR

(b) Since $\vec{\alpha}$ and $\vec{\beta}$ are collinear,

$\vec{\alpha} = k\vec{\beta}$

$(x-2)\vec{a} + \vec{b} = k[(3+2x)\vec{a} - 2\vec{b}]$

Equating their coefficients

For $\vec{b}$, $1 = -2k, k = \frac{-1}{2}$

For $\vec{a}$, $x - 2 = k(3 + 2x)$

or, $x - 2 = \frac{-1}{2}(3 + 2x)$ Substituting $k = -\frac{1}{2}$

or, $2x - 4 = -3 - 2x$

or, $4x = 1 \Rightarrow x = \frac{1}{4}$

17. (a) Find the equation of the plane passing through the intersection of the planes $2x + 2y - 3z - 7 = 0$ and $2x + 5y + 3z - 9 = 0$ such that the intercepts made by the resulting plane on the x-axis and the z-axis are equal. [4]

OR

(b) Find the equation of the lines passing through the point (2, l, 3) and perpendicular to the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$

Ans. Required equation for plane:

$(2x + 2y - 3y - 7) + \lambda(2x + 5y + 3z - 9) = 0$

$(2 + 2\lambda)x + (2 + 5\lambda)y + (3\lambda - 3)z = 7 + 9\lambda$ ...(i)

$\therefore$ x intercept = z intercept

$2 + 2\lambda = 3\lambda - 3$

$\lambda = 5$

Putting $\lambda = 5$ in equation (i);

$12x + 27y + 12z = 52$

OR

(b) Let DR's of the required line be (a, b, c,) and DR's of two given lines are (1, 2, 3) and (– 3, 2, 5)

$\therefore$ Condition of normality,

$a + 2b + 3c = 0$ and $-3a + 2b + 5c = 0$

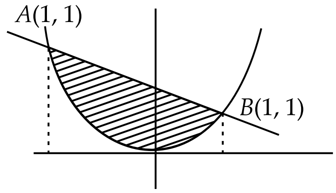
$\frac{a}{2} = \frac{-b}{7} = \frac{c}{4}$

DR's of line (2, – 7, 4) passing through (2, 1, 3)

Equation of line, $\frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$

18. Draw a rough sketch and find the area bounded by the curve $x^2 = y$ and $x + y = 2$. [6]

Ans.



Curve $y = x^2$

Line $x + y = 2$

For Points of intersection, A and B

$x + x^2 = 2$

or, $(x+2)(x-1) = 0$

$x = -2,\ x = 1$

Hence, $y = 4,\ \ y = 1$

$\therefore A(-2,4), B(1,1)$

Area of shaded region

$$A = \underset{\text{Line}}{\int_{-2}^{1} y\,dx} - \underset{\text{Curve}}{\int_{-2}^{1} y\,dx} = \int_{-2}^{1} (2-x)dx - \int_{-2}^{1} x^2\,dx$$

$$A = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$A = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right) = \frac{27}{6} = \frac{9}{2}\text{unit}^2$$

SECTION - C [20 Marks]

19. (a) A company produces a commodity with ₹ 24,000 as fixed cost. The variable cost estimated to be 25% of the total revenue received on selling the product, is at the rate of ₹ 8 per unit. Find the break-even point. [3× 2]

Ans. Fixed Cost = ₹ 24,000

Let no.of units $= x$

Selling price per unit = ₹ 8 $= p(x)$

Revenue function, $R(x) = 8x$

Cost function, $C(x) = 24000 + 25\%$ *of* $8x = 24000 + 2x$

For Break even point,

$R(x) = C(x)$

$8x = 24000 + 2x$

or, $6x = 24000$

or, $x = 4000$

(b) The total cost function for a production is given by $C(x) = \frac{3}{4}x^2 - 7x + 27$.

Find the number of units produced for which *M.C.* = *A.C.* (*M.C.* = Marginal Cost and *A.C.* = Average Cost.)

Ans. Given : $C(x) = \frac{3}{4}x^2 - 7x + 27$

$$AC = \frac{C(x)}{x} = \frac{3}{4}x - 7 + \frac{27}{x}$$

$$MC = \frac{d}{dx}C(x) = \frac{3}{4}.2x - 7$$

$$= \frac{3}{2}x - 7$$

Since, $AC = MC$,

So, $\frac{3}{4}x - 7 + \frac{27}{x} = \frac{3}{2}x - 7$

$$3x^2 + 108 = 6x^2$$

$$x^2 = 36$$

$$x = 6$$

Hence, the required unit will be 6.

(c) If $\bar{x} = 18$, $\bar{y} = 100$, $\sigma_x = 14$, $\sigma_y = 20$ and correlation coefficient $r_{xy} = 0{\cdot}8$, find the regression equation of y on x.

Ans. Given: $\bar{x} = 18$ $\bar{y} = 100$ $\sigma_x = 14$ $\sigma_y = 20$

$r_{xy} = 0.8$

Since,

$$b_{yx} = r_{xy}\frac{\sigma_y}{\sigma_x}$$

$$= 0.8 \times \frac{20}{14} = \frac{8}{7}$$

Regression equation of y on x ;

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 100 = \frac{8}{7}(x - 18)$$

$$8x - 7y + 556 = 0$$

20. (a) The following results were obtained with respect to two variables x and y:

$$\sum x = 15, \sum y = 25, \sum xy = 83, \sum x^2 = 55, \sum y^2 = 135 \text{ and } n = 5$$

(i) Find the regression coefficient b_{xy}.

(ii) Find the regression equation of x on y. [4]

OR

(b) Find the equation of the regression line of y on x, if the observations (x, y) are as follows:

(1, 4), (2, 8), (3, 2), (4, 12), (5, 10), (6, 14), (7, 16), (8, 6), (9, 18).

Also, find the estimated value of y when $x = 14$.

Ans. (i) Given : $\Sigma x = 15, \Sigma y = 25$

$\Sigma xy = 83, \Sigma x^2 = 55$

$\Sigma y^2 = 135, n = 5$

$$b_{xy} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}$$

$$= \frac{83 - \frac{15 \times 25}{5}}{135 - \left(\frac{625}{5}\right)} = \frac{4}{5} = 0.8$$

(ii) $\bar{x} = \frac{15}{5} = 3$, $\bar{y} = \frac{25}{5} = 5$

Regression equation of x on y ;

$$\bar{x} - \bar{y} = b_{xy}\left(\bar{y} - \bar{x}\right)$$

or, $x - 3 = \frac{4}{5}(x - 5)$

$5x - 4y + 5 = 0$

OR

(b)

x	y	x^2	xy
1	4	1	4
2	8	4	16
3	2	9	6
4	12	16	48
5	10	25	50
6	14	36	84
7	16	49	112
8	6	64	48
9	18	81	162
$\Sigma x = 45$	$\Sigma y = 90$	$\Sigma x^2 = 285$	$\Sigma xy = 530$

$\bar{x} = \frac{45}{9} = 5$ $\bar{y} = \frac{90}{9} = 10$

$$b_{yx} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

$$= \frac{530 - \frac{45 \times 90}{9}}{285 - \frac{(45)^2}{9}} = \frac{4}{3}$$

Regression equation of y on x ;

$y - \bar{x} = b_{xy}\,(x - (\bar{y} - \bar{x}))$

$y - 10 = \frac{4}{5}(x - 5)$

$4x - 3y + 10 = 0$

At $x = 14$; $56 - 3y + 10 = 0$

$3y = 66$

$y = 22$

21. **(a)** The cost function of a product is given by $C(x) = \frac{x^3}{3} - 45x^2 - 900x + 36$ where x is the number of units produced. How many units should be produced to minimise the marginal cost?

OR

(b) The marginal cost function of x units of a product is given by $MC = 3x^2 - 10x + 3$. The cost of producing one unit is ₹ 7. Find the total cost function and average cost function. **[4]**

Ans. (a) $C(x) = \frac{x^3}{3} - 45x^2 - 900x + 36$

$MC(x) = x^2 - 90x + 900$

$\frac{d}{dx}(MC) = 2x - 90$

For maxima and minima

$\frac{d}{dx}(MC) = 0$

$2x - 90 = 0 \Rightarrow x = 45$

Now,

$\frac{d^2}{dx}(MC) = 2 > 0$ (+ve)

Hence, at $x = 45$; MC is minimum

OR

(b) $MC = 3x^2 - 10x + 3$

$C(x) = \int MC(dx)$

$= \int (3x^2 - 10x + 3)\,dx$

$C(x) = x^3 - 5x^2 + 3x + c$

Cost of producing 1 unit in ₹ 7 ;

$7 = 1 - 5 + 3 + c$

$\Rightarrow c = 8$

Total cost function, $C(x) = x^3 - 5x^2 + 3x + 8$

Average cost function $AC(x) = \frac{C(x)}{x}$

$= x^2 - 5x + 3 + \frac{8}{x}$

22. A carpenter has 90, 80 and 50 running feet respectively of teak wood, plywood and rosewood which is used to produce product *A* and product *B*. Each unit of product *A* requires 2, 1 and 1 running feet and each unit of product B requires 1, 2 and 1 running feet of teak wood, plywood and rosewood respectively. If product *A* is sold for ₹ 48 per unit and product *B* is sold for ₹ 40 per unit, how many units of product *A* and product *B* should be produced and sold by the carpenter, in order to obtain the maximum gross income?

Formulate the above as a Linear Programming Problem and solve it, indicating clearly the feasible region in the graph. [6]

Ans. Objective function :

Max $Z = 48x + 40y$

Constraints :

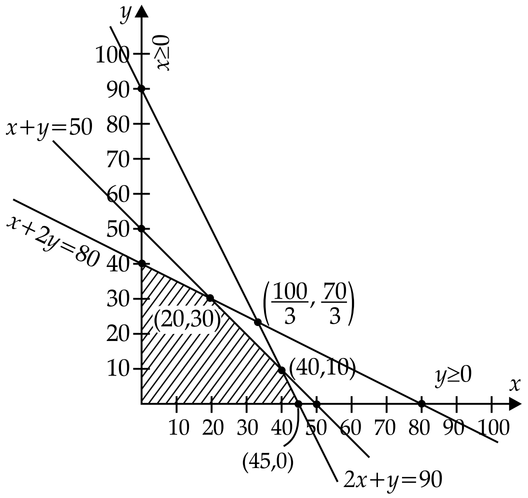
$2x + y \leq 90$

$x + 2y \leq 80$

$x + y \leq 50$

$x \geq 0$

$y \geq 0$



Print(*x*,*y*)	$Z = 48x + 48y$
At (0, 0)	$Z = 0$
(45, 0)	$Z = 48 \times 45 + 40 \times 0 = 2160$
(40, 10)	$Z = 48 \times 40 + 40 \times 10 = 2320$
(20, 30)	$Z = 48 \times 20 + 40 \times 30 = 2160$
(0, 40)	$Z = 48 \times 0 + 40 \times 40 = 1600$

Maximum of Z in 2320 at (40, 10)

$x = 40, \quad y = 10$