ISC Solved Paper 2023

Class-XII

Mathematics

(Maximum Marks : 80)

(Time allowed : Three hours)

*(Candidates are **allowed additional 15 minutes** for **only** reading the paper. They must NOT start writing during this time.)*

This Question Paper consists of three sections A, B and C

*Candidates are required to attempt all questions from **Section A** and all questions*

EITHER** from **Section B** **OR** **Section C.

Section A:** Internal choice has been provided in **two questions of two marks each, two questions of four marks each had two questions of six marks each.

Section B:** Internal choice has been provided in **one question of two marks and one question of four marks.

Section C:** Internal choice has been provided in **one question of two marks and one question of four marks.

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

SECTION - A [65 Marks]

1. In subparts (i) to (x) choose the correct options and in subparts (xi) to (xv), answer the questions as instructed.

(i) A relation R on {1, 2, 3} is given by $R = \{(1, 1), (2, 2), (1, 2), (3, 3), (2, 3)\}$. Then the relation R is: **[1]**

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) Symmetric and Transitive

(ii) If A is a square matrix of order 3, then $|2A|$ is equal to: **[1]**

(a) $2|A|$ (b) $4|A|$
(c) $8|A|$ (d) $6|A|$

(iii) If the following function is continuous at $x = 2$ then the value of k will be: **[1]**

$$f(x) = \begin{cases} 2x+1, & \text{if } x < 2 \\ k, & \text{if } x = 2 \\ 3x-1, & \text{if } x > 2 \end{cases}$$

(a) 2 (b) 3
(c) 5 (d) -1

(iv) An edge of a variable cube in increasing at the rate of 10 cm/s. How fast will the volume of the cube increase if the edge is 5 cm long? **[1]**

(a) 75 cm^3/s (b) 750 cm^3/s
(c) 7500 cm^3/s (d) 1250 cm^3/s

(v) Let $f(x) = x^3$ be a function with domain {0, 1, 2, 3}. Then domain of f^{-1} is: **[1]**

(a) {3, 2, 1, 0} (b) {0, –1, –2, –3}
(c) {0, 1, 8, 27} (d) {0, –1, –8, –27}

(vi) For the curve $y^2 = 2x^3 - 7$, the slope of the normal at (2, 3) is: **[1]**

(a) 4 (b) $\frac{1}{4}$
(c) -4 (d) $\frac{-1}{4}$

(vii) Evaluate: $\int \frac{x}{x^2+1}\,dx$ **[1]**

(a) $2\log(x^2+1)+c$ (b) $\frac{1}{2}\log(x^2+1)+c$
(c) $e^{x^2+1}+c$ (d) $\log x+\frac{x^2}{2}+c$

(viii) The derivative of log x with respect to $\frac{1}{x}$ is: **[1]**

(a) $\frac{1}{x}$ (b) $\frac{-1}{x^3}$
(c) $\frac{-1}{x}$ (d) $-x$

(ix) The interval in which the function $f(x) = 5 + 36x - 3x^2$ increases wilL be: **[1]**

(a) $(-\infty, 6)$ (b) $(6, \infty)$
(c) $(-6, 6)$ (d) $(0, -6)$

(x) Evaluate: $\int_{-1}^{1} x^{17} \cos^4 x \, dx$ **[1]**

(a) ∞ (b) 1
(c) –1 (d) 0

(xi) Solve the differential equation: $\frac{dy}{dx} = \operatorname{cosec} y$ **[1]**

(xii) For what value of k the matrix $\begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix}$ **is a skew symmetric matrix?** **[1]**

(xiii) Evaluate: $\int_0^1 |2x+1| \, dx$ **[1]**

(xiv) Evaluate: $\int \frac{1+\cos x}{\sin^2 x} dx$ **[1]**

(xv) A bag contains 19 tickets, numbered from 1 to 19. Two tickets are drawn randomly in succession with replacement. Find the probability that both the tickets drawn are even numbers. **[1]**

Ans. (i) Option (a) is correct

Explanation: Given set is A = {1, 2, 3} and given relation is $R = \{(1, 1), (2, 2), (1, 2), (3, 3), (2, 3)\}$

R is reflexive as $(1, 1), (2, 2), (3, 3) \in R$

R is not transitive as $(1, 2), (2, 3) \in R \Rightarrow (1, 3) \notin R$

R is not symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$

(ii) Option (c) is correct

Explanation: Given A is a square matrix of order 3

Then, $|2A| = 2^3|A| = 8|A|$

(iii) Option (c) is correct

Explanation: As $f(x)$ is continuous of $x = 2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-1) = 3 \times 2 - 1 = 5$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+1) = 2 \times 2 + 1 = 5$$

$\Rightarrow$ $f(2^-) = f(2^+) = k$ [Since f is continuous at $x = 2$]

$\Rightarrow$ $k = 5$

(iv) Option (b) is correct

Explanation: Let the edge of cube be x cm

Then, $\frac{dx}{dt} = 10$ cm/s ...(i)

Volume of cube, $V = x^3$ cm^3

$\therefore$ $\frac{dV}{dt} = 3x^2\left(\frac{dx}{dt}\right)$ [On differentiating w.r.t t]

$$\left.\frac{dV}{dt}\right|_{x=5 \text{ cm}} = 3(5)^2 \times 10$$

[From eq (i), $\frac{dx}{dt} = 10$ cm/s]

$= 750$ cm^3/s

(v) Option (c) is correct

Explanation:

Given, $f(x) = x^3$

Domain of $f(x) = \{0, 1, 2, 3\}$

$\therefore$ Range of $f(x) = \{0^3, 1^3, 2^3, 3^3\} = \{0, 1, 8, 27\}$

f can be written as,

$f = \{(0, 0), (1, 1), (2, 8), (3, 27)\}$

Now, $f^{-1} = \{(0, 0), (1, 1), (8, 2), (27, 3)\}$

Thus, domain of $f^{-1} = \{0, 1, 8, 27\}$

(vi) Option (d) is correct

Explanation: Given curve is $y^2 = 2x^3 - 7$

On differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 6x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y}$$

Slope of tangent at (2, 3) $= \left.\frac{dy}{dx}\right|_{\text{at } (2, 3)} = \frac{3(2)^2}{3} = 4$

Hence slope of normal at (2, 3) is

$$= \frac{-1}{\text{Slope of tangent at } (2, 3)} = -\frac{1}{4}$$

(vii) Option (b) is correct

Explanation:

$$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} dx = \frac{1}{2} \log(x^2+1) + C$$

(viii) Option (d) is correct

Explanation: Let $u = \log x$ and $v = \frac{1}{x}$

$\therefore$ $\frac{du}{dx} = \frac{1}{x}$

and $\frac{dv}{dx} = -\frac{1}{x^2}$

Thus, $$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x$$

(ix) Option (a) is correct

Explanation:

Given, $f(x) = 5 + 36x - 3x^2$

$\therefore$ $f'(x) = 36 - 6x$

For increasing function, $36 - 6x > 0$

$\Rightarrow 6 - x > 0 \Rightarrow 6 > x$

Hence, $x \in (-\infty, 6)$

(x) Option (d) is correct

Explanation: Let $I = \int_{-1}^{1} x^{17} \cos^4 x \, dx$

Here, $f(x) = x^{17} \cos^4 x$

$f(-x) = -x^{17} \cos^4 x = -f(x)$

Thus, $f(x)$ is an odd function.

We know that,

$$\int_{-a}^{a} f(x)\, dx = 0$$

$$\therefore \quad I = \int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$$

(xi) Given differential equation is

$$\frac{dy}{dx} = \text{cosec } y$$

$$\Rightarrow \quad \frac{dy}{\text{cosec } y} = dx$$

$$\Rightarrow \quad \sin y \, dy = dx$$

$$\Rightarrow \quad \int \sin y \, dy = \int dy$$

$$\Rightarrow \quad -\cos y = x + C$$

$$\Rightarrow \quad x + \cos y + C = 0$$

(xii) Let $A = \begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix}$

Given A is skew symmetric *i.e.*, $A' = -A$

Now, $A' = \begin{bmatrix} 0 & -6 \\ k & 0 \end{bmatrix}$

Thus, $\begin{bmatrix} 0 & -6 \\ k & 0 \end{bmatrix} = -\begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 0 & -6 \\ k & 0 \end{bmatrix} = \begin{bmatrix} 0 & -k \\ 6 & 0 \end{bmatrix}$$

On comparing corresponding elements, we get

$$k = 6$$

(xiii)

$$\int_0^1 |2x+1|\, dx = \int_0^1 (2x+1) dx$$

$$= 2\int_0^1 x\, dx + \int_0^1 1 dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 + [x]_0^1$$

$$= 1 + 1 = 2$$

(xiv)

$$\int \frac{1+\cos x}{\sin^2 x} dx = \int \frac{1+\cos x}{1-\cos^2 x} dx$$

$$= \int \frac{1+\cos x}{(1+\cos x)(1-\cos x)} dx$$

$$= \int \frac{1}{1-\cos x} dx$$

$$= \int \frac{1}{1-\left(1-2\sin^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{2\sin^2 \frac{x}{2}} dx$$

$$= \frac{1}{2}\int \text{cosec}^2 \frac{x}{2} dx$$

$$= \frac{1}{2}\left(-2\cot\frac{x}{2}\right) + C$$

$$= -\cot\frac{x}{2} + C$$

(xv) We have total number of tickets in bag from 1 to 19

$= 1, 2, 3, ..., 19$

$n(S) = 19$

Total even number of tickets in the bag is 2, 4, ..., 18

$n(E) = 9$

Probability (Both the tickets drawn have even numbers)

$$= \frac{9}{19} \times \frac{9}{19}$$

$$= \left(\frac{9}{19}\right)^2$$

2. (i) If $f(x) = [4-(x-7)^3]^{\frac{1}{5}}$ is a real invertible function, then find $f^{-1}(x)$

OR

(ii) Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a function defined by $f(x) = \frac{x-1}{x-2}$ then show that f is a one – one and an onto function [2]

Ans. If $f(x)$ is invertible then it is one-one and onto

We have, $f(x) = [4 - (x - 7)^3]^{1/5}$

<u>**One-One**</u>

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad [4 - (x_1 - 7)^3]^{1/5} = [4 - (x_2 - 7)^3]^{1/5}$$

$$\Rightarrow \quad 4 - (x_1 - 7)^3 = 4 - (x_2 - 7)^3$$

[Taking 5 power both sides]

$$\Rightarrow \quad (x_1 - 7)^3 = (x_2 - 7)^3$$

$$\Rightarrow \quad x_1 - 7 = x_2 - 7$$

[Taking cube root both sides]

$$\Rightarrow \quad x_1 = x_2$$

$\Rightarrow f$ is one-one

<u>**Onto**</u>

Let $y = f(x) = [4 - (x - 7)^3]^{1/5}$

$$\Rightarrow \quad y^5 = 4 - (x - 7)^3$$

$$\Rightarrow \quad (x - 7)^3 = 4 - y^5$$

$$\Rightarrow \quad x - 7 = (4 - y^5)^{1/3}$$

$$\Rightarrow \quad x = (4 - y^5)^{1/3} + 7$$

Thus, $\forall\, y \in R, \exists\, x = \sqrt[3]{4-y^5} + 7 \in R$

$\Rightarrow f$ is onto

Hence, $f^{-1}(y) = x = \sqrt[3]{4-y^5} + 7$

We get $f^{-1}(x)$ if we replace y with x in above equation.

$$f^{-1}(x) = \sqrt[3]{4-x^5} + 7$$

OR

Given, $A = R - \{2\}$, $B = R - \{1\}$

and $f : A \to B$ is defined as $f(x) = \frac{x-1}{x-2}$

One-One

Let $x_1, x_2 \in$ A such that $f(x_1) = f(x_2)$

$$\frac{x_1-1}{x_1-2} = \frac{x_2-1}{x_2-2}$$

$$\Rightarrow (x_1-1)(x_2-2) = (x_2-1)(x_1-2)$$
$$\Rightarrow x_1x_2 - 2x_1 - x_2 + 2 = x_1x_2 - 2x_2 - x_1 + 2$$
$$\Rightarrow -2x_1 - x_2 = -2x_2 - x_1$$
$$\Rightarrow -2x_1 + x_1 = -2x_2 + x_2$$
$$\Rightarrow -x_1 = -x_2$$
$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one

Onto

Let $y \in B = R - \{1\}$, then $y \neq 1$

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now, $f(x) = y$

$$\Rightarrow \frac{x-1}{x-2} = y$$
$$\Rightarrow x - 1 = y(x-2)$$
$$\Rightarrow x - 1 = xy - 2y$$
$$\Rightarrow x(1-y) = 1 - 2y$$
$$\Rightarrow x = \frac{1-2y}{1-y} \in A \qquad (y \neq 1)$$

Thus, for any $y \in B, \exists\, x = \frac{1-2y}{1-y} \in A$ such that

$$f\left(\frac{1-2y}{1-y}\right) = \frac{\left(\frac{1-2y}{1-y}\right)-1}{\left(\frac{1-2y}{1-y}\right)-2}$$
$$= \frac{(1-2y)-(1-y)}{(1-2y)-2(1-y)}$$
$$= \frac{-y}{-1} = y$$

Therefore f is onto.

3. Evaluate the following determinant without expanding. [2]

$$\begin{vmatrix} 5 & 5 & 5 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

Ans. $$\begin{vmatrix} 5 & 5 & 5 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

[Taking 5 common from R_1]

$$= 5\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$[R_3 \to R_3 + R_2]$

$$= 5(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

[Taking $a + b + c$ common from R_3]

$= 5(a + b + c) \times 0$

[R_1 and R_3 are identical]

$= 0$

4. The probability of the event A occurring is $\frac{1}{3}$ and of the event B occurring is $\frac{1}{2}$. If A and B are independent events, then find the probability of neither A nor B occurring. [2]

Ans. Given, $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$

Here, $P(A \cap B) = P(A).P(B)$

[Since, events are independent]

$$= \frac{1}{3}.\frac{1}{2}$$
$$= \frac{1}{6}$$

Thus, P(neither A nor B occurring)

$= 1 - P$(both events occur)

$= 1 - P(A \cap B)$

$$= 1 - \frac{1}{6}$$
$$= \frac{5}{6}$$

5. Solve for x:

$5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ [2]

Ans. Given, $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ ...(i)

We know that,

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$
$$\Rightarrow \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$
$$\therefore 5\tan^{-1}x + 3\left(\frac{\pi}{2} - \tan^{-1}x\right) = 2\pi \quad \text{[From eq (i)]}$$
$$\Rightarrow 5\tan^{-1}x + \frac{3\pi}{2} - 3\tan^{-1}x = 2\pi$$
$$\Rightarrow 2\tan^{-1}x = \frac{\pi}{2}$$
$$\Rightarrow \tan^{-1}x = \frac{\pi}{4}$$
$$\Rightarrow x = \tan\left(\frac{\pi}{4}\right)$$
$$\Rightarrow x = 1$$

6. (i) Evaluate: $\int \cos^{-1}(\sin x)dx$ **[2]**

OR

(ii) If $\int x^5\cos(x^6)dx = k\sin(x^6) + C$, find the value of k.

Ans .(i) $I = \int \cos^{-1}(\sin x)dx$

Let $\cos^{-1}(\sin x) = \theta$

$\Rightarrow \sin x = \cos\theta$

$\Rightarrow \sin x = \sin\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow x = \frac{\pi}{2} - \theta$

$\Rightarrow \theta = \frac{\pi}{2} - x$

$\therefore I = \int \theta\, dx$

$= \int\left(\frac{\pi}{2} - x\right)dx$

$= \frac{\pi}{2}x - \frac{x^2}{2} + C$

(ii) Let $I = \int x^5 \cos(x^6)dx$

Put $x^6 = t$

$\Rightarrow 6x^5dx = dt$

$\Rightarrow x^5dx = \frac{dt}{6}$

$\therefore I = \frac{1}{6}\int \cos t\, dt$

$= \frac{1}{6}\sin t + C$

$= \frac{1}{6}\sin(x^6) + C$

According to question,

$\int x^5 \cos(x^6)dx = k\sin(x^6) + C$

$\therefore \frac{1}{6}\sin(x^6) + C = k\sin(x^6) + C$

On comparing, we get

$k = \frac{1}{6}$

7. If $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$ then prove that $24x^2 - 23x - 12 = 0$ [4]

Ans. $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$

$[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)]$

$\Rightarrow \tan^{-1}\left[\frac{(x-1)(2x+1) + (2x-1)(x+1)}{(x+1)(2x+1) - (x-1)(2x-1)}\right]$

$= \tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \tan^{-1}\left[\frac{(2x^2 - x - 1) + (2x^2 + x - 1)}{(2x^2 + 3x + 1) - (2x^2 - 3x + 1)}\right]$

$= \tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \tan^{-1}\left[\frac{4x^2 - 2}{6x}\right] = \tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \frac{4x^2 - 2}{6x} = \frac{23}{36}$

$\Rightarrow \frac{2x^2 - 1}{3x} = \frac{23}{36}$

$\Rightarrow 36(2x^2 - 1) = 23(3x)$

$\Rightarrow 12(2x^2 - 1) = 23x$

$\Rightarrow 24x^2 - 23x - 12 = 0$ **Hence Proved**

8. If $y = e^{ax}\cos bx$, then prove that

$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Ans.Given, $y = e^{ax}\cos bx$ ...(i)

$\frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx$ ...(ii)

$\therefore \frac{dy}{dx} = ay - be^{ax}\sin bx$ [from eq. (i)]

$\frac{d^2y}{dx^2} = a\frac{dy}{dx} - b(ae^{ax}.\sin bx + be^{ax}\cos bx)$

$\frac{d^2y}{dx^2} = a\frac{dy}{dx} - a\left(ay - \frac{dy}{dx}\right) - b^2y$

[from eq. (i)]

$\frac{d^2y}{dx^2} = a\frac{dy}{dx} - a^2y + a\frac{dy}{dx} - b^2y$

$\therefore \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Hence Proved

9. (i) In a company, 15% of the employees are graduates and 85% of the employees are non-graduates. As per the annual report of the company, 80% of the graduate employees and 10% of the non-graduate employees are in the Administrative position. Find the probability that an employee selected at random from those working in administrative position will be a graduate. **[4]**

OR

(ii) A problem in Mathematics is given to three students A, B and C. Their chances of solving the problem are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that

(a) exactly two students will solve the problem.

(b) at least two of them will solve the problem.

Ans. Probability of graduate employees

$$= P(E_1) = \frac{15}{100}$$

Probability of non-graduate employees

$$= P(E_2) = \frac{85}{100}$$

Probability of graduate employees in Administrative position

$$= P\left(\frac{A}{E_1}\right) = \frac{80}{100}$$

Probability of non-graduate employees in Administrative position

$$= P\left(\frac{A}{E_2}\right) = \frac{10}{100}$$

By Bayes' Theorem

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{80}{100}\times\frac{15}{100}}{\frac{80}{100}\times\frac{15}{100}+\frac{85}{100}\times\frac{10}{100}}$$

$$= \frac{1200}{1200+850}$$

$$= \frac{1200}{2050}$$

$$= 0.5853$$

$$= 0.59$$

OR

Given, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

$P(\bar{A}) = \frac{1}{2}$, $P(\bar{B}) = \frac{2}{3}$, $P(\bar{C}) = \frac{3}{4}$,

(a) Probability that exactly two students will solve the problem

$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$

$= P(A).P(B).P(\bar{C}) + P(A).P(\bar{B}).P(C) + P(\bar{A}).P(B).P(C)$

$$= \frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}+\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}$$

$$= \frac{3+2+1}{2\times3\times4}$$

$$= \frac{6}{24} = \frac{1}{4}$$

(b) Probability that atleast two of them will solve the problem

= Probability that exactly two students will solve the problem + Probability that all solve the problem

$= \frac{1}{4} + P(A).P(B).P(C)$ [from (a)]

$$= \frac{1}{4}+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4} = \frac{1}{4}+\frac{1}{24} = \frac{7}{24}$$

10. (i) Solve the differential equation:

$(1 + y^2)dx = (\tan^{-1}y - x)dy$

OR

(ii) Solve the differential equation:

$(x^2 - y^2)dx + 2xydy = 0$ **[4]**

Ans. (i) We have, $(1 + y^2)dx = (\tan^{-1} y - x)dy$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is of the form $\frac{dx}{dy} + Px = Q$

$$\therefore \quad IF = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$$\text{I.F.} = e^{\int\frac{1}{1+y^2}dx} = e^{\tan^{-1} y}$$

$$\text{I.F.} \times x = \int I.F.\times Q dy$$

$$\Rightarrow e^{\tan^{-1}} y.x = \int e^{\tan^{-1}} y.\frac{\tan^{-1} y}{1+y^2}dy$$

$$\Rightarrow e^{\tan^{-1}} y.x = \int \underset{\text{II}}{e^t}.\underset{\text{I}}{t}.dt$$

$$\Rightarrow e^{\tan^{-1}} y.x = t.e^t - \int 1.e^t dy$$

$$= t.e^t - e^t + c$$

$$= e^t(t - 1) + c$$

$$\Rightarrow e^{\tan^{-1}} y.x = e^{\tan^{-1}} y(\tan^{-1} y - 1) + c$$

OR

Given, $(x^2 - y^2)dx + 2xy\, dy = 0$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx$, $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2xvx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(v^2-1)}{x^2 2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(1+v^2)}{2v}$$

$$\Rightarrow \frac{2v}{1+v^2}dv = \frac{-1}{x}dx$$

by Superable Method on integrating both side

$$\Rightarrow \int\frac{2v}{1+v^2}dx = \int -\frac{1}{x}dx$$

$$\Rightarrow \log(1 + v^2) = -\log x + \log c$$

$$\Rightarrow \quad \log(1+v^2) = \log\frac{c}{x}$$

$$\Rightarrow \quad (1+v^2) = \frac{c}{x}$$

$$\Rightarrow \quad x(1+v^2) = c$$

$$\Rightarrow \quad \frac{x(x^2+y^2)}{x^2} = c$$

$$\Rightarrow \quad x^2+y^2 = cx$$

11. Use matrix method to solve the following system of equations. **[6]**

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Ans. Given system of equations is

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2$$

Let $\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$

The system of equations become

$$2u + 3v + 10w = 4$$
$$4u - 6v + 5w = 1$$
$$6u + 9v - 20w = 2$$

Writing equation as $AX = B$

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Hence, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$ $X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$|A| = 2\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$$= 2(120-45) - 3(-80-30) + 10(36+36)$$
$$= 2(75) - 3(-110) + 10(72)$$
$$= 150 + 330 + 720$$
$$= 1200$$

$\therefore \quad |A| \neq 0$

So, the system of equation is consistent and has unique solution

Now, $\quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -110 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now, $\quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$$= \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -110 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Thus, $\quad X = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -110 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 140 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

Hence, $u = \frac{1}{2}$, $v = \frac{1}{3}$, and $w = \frac{1}{5}$

Thus, $x = 2$, $y = 3$ and $z = 5$

12. (i) Prove that the semi-vertical angle of the right circular cone of given volume and least curved area is $\cot^{-1}\sqrt{2}$. **[6]**

OR

(ii) A running track of 440 m is to be laid out enclosing a football field. The football field is in the shape of a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the length of its sides. Also calculate the area of the football field.

Ans. Let h, r and α be the height, radius and semi-vertical angle of the right angled triangle.

We know, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad h = \frac{3V}{\pi r^2}$$

Also, slant height, $l = \sqrt{h^2+r^2}$

Curved surface area is given by

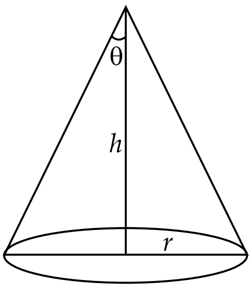
$$A = \pi rl = \pi r\sqrt{r^2+h^2}$$

$$A = \pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}$$

$$\therefore \quad A = \sqrt{\pi^2 r^4+\frac{9V^2}{r^2}}$$

Differentiating the above w.r.t. r, we get

$$\frac{dA}{dr} = \frac{1}{2\sqrt{\pi^2 r^4+\frac{9V^2}{r^2}}}\left(4\pi^2 r^3-\frac{18V^2}{r^3}\right)$$



A is maximum or minimum, when $\frac{dA}{dr} = 0$

$$\therefore \frac{1}{2\sqrt{\pi^2 r^4+\frac{9V^2}{r^2}}}\left(4\pi^2 r^3-\frac{18V^2}{r^3}\right) = 0$$

$\Rightarrow \quad 4\pi^2 r^3 - 18V^2 r^{-3} = 0$

$\Rightarrow \quad 4\pi^2 r^3 = 18V^2 r^{-3}$

$\Rightarrow \quad 4\pi^2 r^3 = 18\left(\frac{1}{3}\pi r^2 h\right)^2 r^{-3}$

$\Rightarrow \quad \frac{h}{r} = \sqrt{2}$

Thus, $\cot\theta = \sqrt{2}$

Hence, semi-vertical angle, $\theta = \cot^{-1}\sqrt{2}$

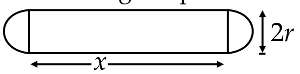
Also, for $r < \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$, $\frac{dA}{dr} < 0$

and for $r > \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$, $\frac{dA}{dr} > 0$

So, curved surface area for $r^3 = \frac{3V}{\pi\sqrt{2}}$ or $V = \frac{\pi r^3\sqrt{2}}{3}$ is the least.

OR

Here, area of rectangular portion $= 2rx$



and perimeter of track $= 2x + 2\pi r$

Given perimeter of track $= 440$ m

$\therefore \quad 2x + 2\pi r = 440 \quad$ …(i)

So, area of rectangular portion, $A = r(440 - 2\pi r)$ [from (i)]

$A = 440r - 2\pi r^2$

Differentiating A w.r.t. r, we get

$$\frac{dA}{dr} = 440 - 4\pi r$$

Area to be maximum,

$$\frac{dA}{dr} = 0$$

$\therefore \quad 440 - 4\pi r = 0$

$\Rightarrow \quad r = \frac{110}{\pi}$

Now, $\frac{d^2A}{dr^2} = -4\pi < 0$

From (i), we get

$2x = 440 - 2\pi r$

$= 440 - 2\pi\left(\frac{110}{\pi}\right) \quad \left[\because r = \frac{110}{\pi}\right]$

$= 440 - 220$

$= 220$

$\therefore \quad x = 110$

Hence sides of rectangle are $2r = \frac{220}{\pi}$ m and $x = 110$ m.

Area of football field = Area of rectangle + 2(area of semicircle)

$= x \times 2r + 2\left(\frac{\pi r^2}{2}\right)$

[r = radius of semi-circle]

$= 110 \times \frac{220}{\pi} + \pi\left(\frac{110}{\pi}\right)^2$

$= \frac{24200}{\pi} + \frac{12100}{\pi}$

$= \frac{36300}{\pi}$

$= \frac{36300}{22} \times 7$

$= 1650 \times 7$

$= 11{,}550 \text{ m}^2$

13. (i) Evaluate: $\int \frac{3e^{2x}-2e^x}{e^{2x}+2e^x-8}\,dx$ **[6]**

OR

(ii) Evaluate: $\int \frac{2}{(1-x)(1+x^2)}\,dx$

Ans. (i) Let $I = \int \frac{3e^{2x}-2e^x}{e^{2x}+2e^x-8}dx$

let $e^x = t$

$\Rightarrow \quad e^x dx = dt$

$\Rightarrow \quad dx = \frac{dt}{t}$

$\therefore \quad I = \int \frac{3t^2-2t}{t^2+2t-8}\frac{dt}{t}$

$$= \int \frac{3t-2}{t^2+2t-8}dt$$

Now, $\frac{3t-2}{t^2+2t-8} = \frac{3t-2}{(t+4)(t-2)}$

So, $\frac{3t-2}{(t+4)(t-2)} = \frac{A}{(t+4)} + \frac{B}{(t-2)}$

$$3t-2 = A(t-2) + B(t+4)$$
$$3t-2 = (A+B)t + (4B-2A)$$

On comparing, we get

$$A + B = 3 \quad \ldots(i)$$

and $4B - 2A = -2$

or $2B - A = -1 \quad \ldots(ii)$

On solving eqs. (i) & (ii), we get

$$A = \frac{7}{3} \text{ and } B = \frac{2}{3}$$

$\therefore \quad \frac{3t-2}{t^2+2t-8} = \frac{7}{3}\frac{1}{(t+4)} + \frac{2}{3}\frac{1}{t-2}$

$\therefore \quad I = \frac{7}{3}\int\frac{1}{t+4}dt + \frac{2}{3}\int\frac{1}{t-2}dt$

$$= \frac{7}{3}\log|t+4| + \frac{2}{3}\log|t-2| + C$$

$$= \frac{7}{3}\log|e^x+4| + \frac{2}{3}\log|e^x-2| + C$$

[substituting $t = e^x$]

OR

(ii) Let $I = \int\frac{2}{(1-x)(1+x^2)}$

We can write integrand as

$$\frac{2}{(1-x)(1+x^2)} = \frac{2}{-(x-1)(1+x^2)}$$
$$= \frac{-2}{(x-1)(1+x^2)}$$

Applying partial fraction,

$$\frac{-2}{(x-1)(1+x^2)} = \frac{A}{(x-1)} + \frac{Bx+C}{(1+x^2)}$$

$$\frac{-2}{(x-1)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(x-1)}{(x-1)(1+x^2)}$$

$$-2 = A(1+x^2) + (Bx+C)(x-1) \quad \ldots(i)$$

Putting $x = 1$, we get

$$-2 = 2A + 0$$

$\Rightarrow \quad A = -1$

Putting $x = 0$, we get

$$-2 = A + (-C)$$

$\Rightarrow \quad -2 = -1 - C \quad [\because A = -1]$

$\Rightarrow \quad C = 1$

Putting $A = -1$, $C = 1$ in eq (i), we get

$$B = 1$$

So, $\frac{-2}{(x-1)(1+x^2)} = \frac{-1}{(x-1)} + \frac{x+1}{(x^2+1)}$

$\therefore \quad I = \int -\frac{1}{(x-1)}dx + \int\frac{x+1}{x^2+1}dx$

$$= -\int\frac{1}{(x-1)}dx + \int\frac{x}{x^2+1}dx + \int\frac{1}{x^2+1}dx$$

$$= -\int\frac{1}{(x-1)}dx + I_1 + \int\frac{1}{x^2+1}dx$$

where $I_1 = \int\frac{x}{x^2+1}dx$

let $x^2 + 1 = t$

$\Rightarrow \quad 2x\,dx = dt$

$\Rightarrow \quad xdx = \frac{dt}{2}$

$\therefore \quad I_1 = \frac{1}{2}\int\frac{dt}{t}$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|x^2+1| + C_1 \quad \ldots(ii)$$

Thus, $I = -\int\frac{1}{(x-1)}dx + \int\frac{x}{x^2+1}dx + \int\frac{1}{x^2+1}dx$

$$= -\log|x-1| + \frac{1}{2}\log|x^2+1| + \tan^{-1}x + C_1 + C_2$$

[from eq. (ii)]

$$= -\log|x-1| + \frac{1}{2}\log|x^2+1| + \tan^{-1}x + C$$

$[C = C_1 + C_2]$

14. A box contains 30 fruits, out of which 10 are rotten. Two fruits are selected at random one by one without replacement from the box. Find the probability distribution of the number of unspoiled fruits. Also find the mean of the probability distribution. [6]

Ans. Total number of fruits = 30

Number of rotten (spoiled) fruits = 10

Number of unspoiled fruits = 20

Probability of rotten fruits $P(\text{rotten}) = \frac{10}{30}$

Probability of unspoiled fruits $P(\text{unspoiled}) = \frac{20}{30}$

Let X be the random variable of number of unspoiled fruit.

So, $X = 0, 1, 2$

Two fruit can be drawn in 30 fruits

$P(X = 0)$ = Two fruits are spoiled fruits

$$= \frac{{}^{10}C_2}{{}^{30}C_2} = \frac{10\times9\times2}{2\times30\times29}$$

$$= \frac{9}{87}$$

$P(X = 1)$ = 1 fruit is unspoiled and 1 fruit is spoiled

$$= \frac{{}^{20}C_1 \cdot {}^{10}C_1}{{}^{30}C_2} = \frac{20\times10\times2}{30\times29}$$

$$= \frac{40}{87}$$

$P(X = 2)$ = Two fruits are unspoiled

$$= \frac{^{20}C_2}{^{30}C_2} = \frac{20\times19\times2}{30\times29\times2}$$

$$= \frac{38}{87}$$

Mean of probability distribution

$$= \sum X_i P(X_i)$$
$$= X_0P(X_0) + X_1P(X_1) + X_2P(X_2)$$
$$= 0\times\frac{9}{87}+1\times\frac{40}{87}+2\times\frac{38}{87}$$
$$= \frac{116}{87} = 1.33$$

SECTION - B [15 Marks]

15. In subparts (i) and (ii) choose the correct options and in subparts (iii) to (v), answer the questions as instructed. **[5]**

(i) If $|\vec{a}|=3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector then the angle between $\vec{a}$ and $\vec{b}$ will be:

(a) $\frac{\pi}{6}$ **(b)** $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ **(d)** $\frac{\pi}{2}$

(ii) The distance of the point $2\hat{i}+\hat{j}-\hat{k}$ from the plane $\vec{r}.(\hat{i}-2\hat{j}+4\hat{k})=9$ will be:

(a) 13 **(b)** $\frac{13}{\sqrt{21}}$

(c) 21 **(d)** $\frac{21}{\sqrt{13}}$

(iii) Find the area of the parallelogram whose diagonals are $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.

(iv) Write the equation of the plane passing through the point (2, 4, 6) and marking equal intercepts on the coordinate axes.

(v) If the two vectors $3\hat{i}+a\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}+8\hat{k}$ are perpendicular to each other, then find the value of a.

Ans. (i) Option (b) is correct

Explanation: Given, $|\vec{a}|=3$, $|\vec{b}|=\frac{\sqrt{2}}{3}$

and $|\vec{a}\times\vec{b}| = 1$

We know that,

$$|\vec{a}\times\vec{b}| = \left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right|$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.1$$

$$\Rightarrow \quad 1 = 3\times\frac{\sqrt{2}}{3}\sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

(ii) Option (b) is correct

Explanation: We know that

$$\text{distance} = \frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$$

Given, $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$, $\vec{n}=\hat{i}-2\hat{j}+4\hat{k}$, $d=9$

$$\therefore \quad \text{distance} = \frac{|(2\hat{i}+\hat{j}-\hat{k}).(\hat{i}-2\hat{j}+4\hat{k})-9|}{|\hat{i}-2\hat{j}+4\hat{k}|}$$

$$= \frac{|2-2-4-9|}{\sqrt{1+4+16}}$$

$$= \frac{13}{\sqrt{21}}$$

(iii) Let $\vec{d_1}=\hat{i}-3\hat{j}+\hat{k}$ and $\vec{d_2}=\hat{i}+\hat{j}+\hat{k}$

$$\text{Area of parallelogram} = \frac{1}{2}|\vec{d_1}\times\vec{d_2}|$$

$$\vec{d_1}\times\vec{d_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 1 & 1 & 1\end{vmatrix}$$

$$= \hat{i}(-3-1)-\hat{j}(1-1)+\hat{k}(1+3)$$

$$= -4\hat{i}+4\hat{k}$$

$$\therefore \quad |\vec{d_1}\times\vec{d_2}| = \sqrt{(-4)^2+(4)^2}$$

$$= \sqrt{16+16} = \sqrt{32}$$

Thus, area of parallelogram

$$= \frac{1}{2}\sqrt{32}\text{ unit}^2$$

$$= \sqrt{8}\text{ unit}^2$$

(iv) The equation of the plane intercepts on the coordinate axes area a, b and c is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

Given, $a = b = c$

$\therefore \quad \frac{x}{a}+\frac{y}{a}+\frac{z}{a} = 1$

$\Rightarrow \quad x + y + z = a$

This plane passes through point (2, 4, 6).

$\therefore \quad 2 + 4 + 6 = a$

$\Rightarrow \quad a = 12$

Thus, required equation of plane is $x + y + z = 12$.

(v) For perpendicular vectors $\vec{a}.\vec{b} = 0$

$\therefore (3\hat{i}+\alpha\hat{j}+\hat{k}).(2\hat{i}-\hat{j}+8\hat{k}) = 0$

$\Rightarrow \quad 6-\alpha+8 = 0$

$\Rightarrow \quad \alpha = 14$

16. **(i)** A (1, 2, –3) and B (–1, –2, 1) are the end points of a vector $\vec{AB}$ then find the unit vector in the direction of $\vec{AB}$. **[2]**

OR

(ii) If $\hat{a}$ is unit vector and $(2\vec{x}-3\hat{a}).(2\vec{x}+3\hat{a}) = 91$, find the value of $|\vec{x}|$.

Ans. (i) We have, $A(1, 2, –3)$ and $B(–1, –2, 1)$

$$\vec{AB} = (-1-1)\hat{i}+(-2-2)\hat{j}+(1+3)\hat{k}$$

$$\therefore \quad \vec{AB} = -2\hat{i}-4\hat{j}+4\hat{k}$$

Unit vector in direction of $\vec{AB}$

$$= \frac{-2\hat{i}-4\hat{j}+4\hat{k}}{\sqrt{(-2)^2+(-4)^2+(4)^2}}$$

$$= \frac{-2(\hat{i}+2\hat{j}-2\hat{k})}{\sqrt{4+16+16}}$$

$$= \frac{-2(\hat{i}+2\hat{j}-2\hat{k})}{6}$$

$$= -\frac{1}{3}(\hat{i}+2\hat{j}-2\hat{k})$$

OR

(ii) Given, $(2\vec{x}-3\hat{a}).(2\vec{x}+3\hat{a}) = 91$

$\Rightarrow \quad 4|\vec{x}|^2+6\vec{x}.\hat{a}-6\hat{a}.\vec{x}-9|\hat{a}|^2 = 91$

$\Rightarrow \quad 4|\vec{x}|^2+6.\vec{x}.1-6.1.\vec{x}-9.1 = 91 \quad [\because \hat{a}=1]$

$\Rightarrow \quad 4|\vec{x}|^2 = 100$

$\Rightarrow \quad |\vec{x}|^2 = 25$

$\Rightarrow \quad |\vec{x}| = 5$

17. **(i)** Find the equation of the plane passing through the point (1,1,–1) and perpendicular to the planes $x + 2y + 3z = 7$ and $2x – 3y + 4z = 0$. **[4]**

OR

(ii) A line passes through the point (2, –1, 3) and is perpendicular to the lines

$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and

$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation.

Ans. Let the equation of plane passing through point (1, 1, – 1) be

$a(x – 1) + b(x – 1) + c(x + 1) = 0$ …(i)

Eq (i) is perpendicular to the plane $x + 2y + 3z – 7 = 0$

$\therefore \quad 1.a + 2.b + 3.c = 0$

$\Rightarrow \quad a + 2b + 3c = 0$ …(ii)

Again eq (i) is perpendicular to plane $2x – 3y + 4z = 0$

$\therefore \quad 2.a – 3.b + 4.c = 0$

$\Rightarrow \quad 2a – 3b + 4c = 0$ …(iii)

On solving eqs (ii) and (iii), we get

$$\frac{a}{8+9} = \frac{b}{6-4} = \frac{c}{-3-4}$$

$$\Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{-7} = k$$

$\Rightarrow a = 17k, b = 2k$ and $c = – 7k$

Putting the values of a, b and c in eq (i), we get

$17k(x – 1) + 2k(y – 1) – 7k(z + 1) = 0$

$\Rightarrow \quad 17(x – 1) + 2(y – 1) – 7(z + 1) = 0$

$\Rightarrow \quad 17x + 2y – 7z – 17 – 2 – 7 = 0$

$\Rightarrow \quad 17x + 2y – 7z – 26 = 0$

OR

Let eq. of required line passing Through (2, –1, 3) is

$$\vec{r} = (2\vec{i}-\vec{j}-3\vec{k})+\mu(\vec{i}+2\vec{j}+2\vec{k})$$

eq. (1) is perpendicular to

$$\vec{r} = (\vec{i}-\vec{j}-\vec{k})+\lambda(2\vec{i}+\vec{j}+\vec{k})$$

$\therefore \quad 2x – 2y + 2 = 0$

Again eq (i) is perpendicular to

$$\vec{r} = (2\vec{i}-\vec{j}-3\vec{k})+\pi(\vec{i}+2\vec{j}+2\vec{k})$$

$\therefore \quad k + 2y + 22 = 0$

On solving eq. (2) and (3) we get

$$\frac{x}{-4-2} = \frac{y}{1-4} = \frac{z}{4-2}$$

$$\frac{x}{-6} = \frac{y}{-3} = \frac{z}{2} = k$$

$\Rightarrow x = – 6k, y = 3k$ and $z = 2k$

Putting the values of x, y and z k eq

we get $\vec{r} = (2\vec{i}-3\vec{j}+3\vec{k})+\lambda_1(-6\vec{i}-3\vec{j}+3\vec{k})$

18. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y – 2$. **[4]**

Ans. Given curve $x^2 = 4y$ ... (i) represents an upward parabola with vertex (0, 0) and axis along y-axis

Given equation of line is $x = 4y - 2$ ...(ii)

On solving eqs (i) & (ii), we get

$$x^2 = x + 2$$

$\Rightarrow$ $x^2 - x - 2 = 0$

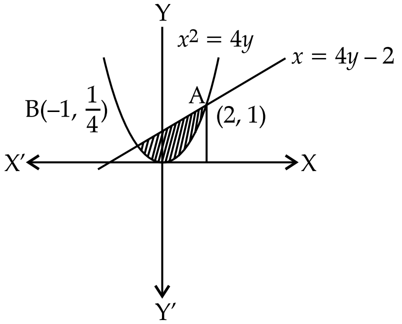
$\Rightarrow$ $(x - 2)(x + 1) = 0$

$\Rightarrow$ $x = 2, -1$

when $x = 2, y = 1$

and $x = -1, y = \frac{1}{4}$

Thus, line meets the parabola at the points A(2, 1) and $B\left(-1, \frac{1}{4}\right)$.



Required area = (Area under the line $x = 4y - 2$) – (Area under the parabola $x^2 = 4y$)

$$= \int_{-1}^{2}\left(\frac{x+2}{4}\right)dx - \int_{-1}^{2}\frac{x^2}{4}dx$$

[From eq. (ii), $y = \frac{x+2}{4}$ and from eq. (i), $y = \frac{x^2}{4}$]

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left\{\frac{2^2}{2} + 2(2)\right\} - \left\{\frac{(-1)^2}{2} + 2(-1)\right\}\right] - \frac{1}{12}(2^3 - (-1)^3)$$

$$= \frac{1}{4}\left(6 + \frac{3}{2}\right) - \frac{1}{12} \times 9$$

$$= \frac{15}{8} - \frac{9}{12}$$

$$= \frac{9}{8} \text{ sq. unit}$$

SECTION - A [15 Marks]

19. In subparts (i) and (ii) choose the correct options and in subparts (iii) to (v), answer the questions as instructed. [5]

(i) If the demand function is given by $p = 1500 - 2x - x^2$ then find the marginal revenue when $x = 10$

(a) 1160 **(b)** 1600

(c) 1100 **(d)** 1200

(ii) If the two regression coefficients are 0.8 and 0.2, then the value of coefficient of correlation r will be:

(a) ± 0.4 **(b)** ± 0.16

(c) 0.4 **(d)** 0.16

(iii) Out of the two regression lines $x + 2y - 5 = 0$ and $2x + 3y = 8$, find the line of regression of y on x.

(iv) The cost function $C(x) = 3x^2 - 6x + 5$. Find the average cost when $x = 2$.

(v) The fixed cost of a product is ₹ 30,000 and its variable cost per unit is ₹ 800. If the demand function is $p(x) = 4500 - 100x$, find the break-even values.

Ans. (i) Option (a) is correct.

***Explanation*:** Given, $p = 1500 - 2x - x^2$

Revenue function, $R = px$

$R = 1500x - 2x^2 - x^3$

Marginal revenue $= \frac{dR}{dx}$

$= \frac{d}{dx}(1500x - 2x^2 - x^3)$

$= 1500 - 4x - 3x^2$

Marginal revenue at $x = 10$ is

$= 1500 - 4(10) - 3(10)^2$

$= 1500 - 40 - 300$

$= 1160$

(ii) Option (c) is correct.

***Explanation*:** $r = \sqrt{0.8 \times 0.2}$

$= \sqrt{0.16}$

$= \sqrt{(0.4)^2}$

$= 0.4$

Here, correlation coefficient will be positive because both the coefficients are positive.

(iii) $x + 2y - 5 = 0$ ...(i)

$2x + 3y = 8$ ...(ii)

Let eq (i), be on x and eq. (ii) be x on

Slope of eq (i) $= -\frac{1}{2}$

Slope of eq (ii) $= -\frac{2}{3}$

$\Rightarrow$ $b_{yx} = -\frac{1}{2}, \& + \frac{1}{b_{xy}} = \frac{-1}{3}$

$\Rightarrow$ $b_{yx} = -\frac{1}{2}, \&\ b_{xy} = \frac{-3}{2}$

Since both b_{yx} and b_{xy} are of since sign and

$$b_{yx} \times b_{xy} = -\frac{1}{2} \times \frac{-3}{2} = \frac{3}{4} < 1$$

∴ Our assumption is true

Hence eq. (i) *i.e.*, $x + 2y - 5 = 0$ is a line of regression of y on x.

From eq (ii), the regression line of y on x is

(iv) Given, $C(x) = 3x^2 - 6x + 5$

$$\text{Average cost} = \frac{C(x)}{x}$$

$$= 3x - 6 + \frac{5}{x}$$

$$(\text{Average cost})_{\text{at } x=2} = 3(2) - 6 + \frac{5}{2}$$

$$= \frac{5}{2} = 2.5$$

(v) Total cost = fixed cost + variable cost

$C(x) = ₹\ 30{,}000 + ₹\ 800x$

where x = total unit

Also, revenue function, $R(x) = p.x$

$= (4500 - 100x)x$

$[\because \text{Given } p = 4500 - 100x]$

$= 4500x - 100x^2$

Profit function $P(x) = R(x) - C(x)$

$= 4500x - 100x^2 - 30000 - 800x$

$= -100x^2 + 3700x - 30000$

At break even point, $P(x) = 0$

$\therefore -100x^2 + 3700x - 30000 = 0$

or, $x^2 - 37x + 300 = 0$

$$x = \frac{+37 \pm \sqrt{(-37)^2 - 4(1)(300)}}{2 \times 1}$$

$$= \frac{37 \pm \sqrt{1369 - 1200}}{2}$$

$$= \frac{37 \pm 13}{2}$$

$$= \frac{37+13}{2} \text{ and } \frac{37-13}{2}$$

$$= \frac{50}{2} \text{ and } \frac{24}{2}$$

$= 25$ and 12

So, break even values are 25 and 12.

20. (i) The total cost function for x units is given by $C(x) = \sqrt{6x+5} + 2500$. Show that the marginal cost decreases as the output x increases. **[2]**

OR

(ii) The average revenue function is given by $AR = 25 - \frac{x}{4}$.

Find total revenue function and marginal revenue function.

Ans. (i) Given, $C(x) = \sqrt{6x+5} + 2500$

$$MC = \frac{dC}{dx}$$

$$= \frac{d}{dx}[\sqrt{6x+5} + 2500]$$

or, $$MC = \frac{1}{2}(6x+5)^{-1/2}(6)$$

or, $$MC = \frac{3}{\sqrt{6x+5}}$$

Now, put $x = 2$,

$$MC = \frac{3}{\sqrt{12+5}} = \frac{3}{\sqrt{17}}$$

$$= \frac{3}{4.12} = 0.72$$

Put $x = 3$,

$$MC = \frac{3}{\sqrt{18+5}} = \frac{3}{\sqrt{23}}$$

$$= \frac{3}{4.79} = 0.62$$

So, it is clear, as we increase output x, MC decreases.

OR

(ii) Given, average revenue = $AR = 25 - \frac{x}{4}$

Since, $$AR = \frac{R}{x}$$

$$= \frac{p.x}{x} = p$$

$$\therefore \quad p = 25 - \frac{x}{4}$$

$$\text{Total revenue } R(x) = p.x = 25x - \frac{x^2}{4}$$

$$\text{Marginal revenue, MR} = \frac{d}{dx}R(x)$$

$$= \frac{d}{dx}\left(25x - \frac{x^2}{4}\right)$$

$$= 25 - \frac{x}{2}$$

21. Solve the following Linear Programming Problem graphically. **[4]**

Maximise $Z = 5x + 2y$ subject to:

$x - 2y \leq 2$,

$3x + 2y \leq 12$,

$-3x + 2y \leq 3$,

$x \geq 0, y \geq 0$

Ans. Given LPP is

Max $z = 5x + 2y$

Subject to: $x - 2y \leq 2$

$3x + 2y \leq 12$

$-3x + 2y \leq 3$

$x \geq 0, y \geq 0$

Converting the inequations into equations, we get

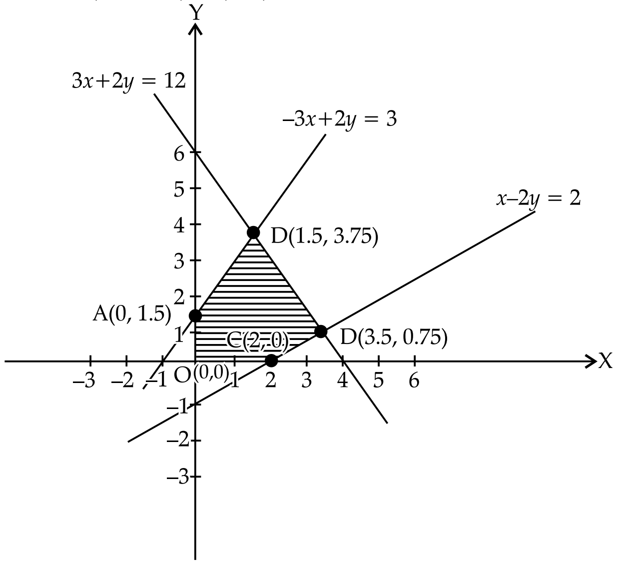
$x - 2y = 2$ ...(i)

$3x + 2y = 12$ ...(ii)

$$-3x + 2y = 3 \quad \ldots\text{(iii)}$$

$$x = 0, y = 0 \quad \ldots\text{(iv)}$$

On plotting the above set of equation, we get the corner points as A(0, 1.5), B(3.5, 0.75), C(2, 0), D(1.5, 3.75), O(0, 0).



The value of the objective function are:

Point (x, y)	$z = 5x + 2y$
A(0, 1.5)	$5 \times 0 + 2 \times 1.5 = 3$
B(3.5, 0.75)	$5 \times 3.5 + 2 \times 0.75 = 19$ (max)
C(2, 0)	$5 \times 2 + 2 \times 0 = 10$
D(1.5, 3.75)	$5 \times 1.5 + 2 \times 3.75 = 15$
O(0, 0)	$5 \times 0 + 2 \times 0 = 0$

So, maximum value of z is 19.

22. (i) The following table shows the Mean, the Standard Deviation and the coefficient of correlation of two variables x and y. **[4]**

Series	x	y
Mean	8	6
Standard deviation	12	4
Coefficient of correlation	0.6	

Calculate:

(a) the regression coefficient b_{xy} and b_{yx}

(b) the probable value of y when $x = 20$

OR

(ii) An analyst analysed 102 trips of a travel company. He studied the relation between travel expenses (y) and the duration (x) of these trips. He found that the relation between x and y was linear. Given the following data, find the regression equation of y on x.

$\Sigma x = 510, \Sigma y = 7140, \Sigma x^2 = 4150, \Sigma y^2 = 740200, \Sigma xy = 54900$

Ans.(i) Given, $r = 0.6$

Mean of $x = \bar{x} = 8$

Mean of $y = \bar{y} = 6$

S.D. of $x = \sigma_x = 12$

S.D. of $y = \sigma_y = 4$

(i) $$b_{xy} = \frac{r\sigma_x}{\sigma_y} = \frac{0.6\times 12}{(4)}$$

$$= \frac{0.6\times 12}{4} = 1.8$$

$$b_{yx} = \frac{r\sigma_y}{\sigma_x} = \frac{0.6\times 4}{(12)}$$

$$= \frac{0.6\times 4}{12} = 0.2$$

(ii) Regression line y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 6 = 0.2\,(x - 8)$$

$$y = 0.2x - 1.6 + 6$$

$$= 0.2x + 4.4$$

at $x = 20$

$$= 0.2 \times 20 + 4.4$$

$$= 4 + 4.4$$

$$y = 8.4$$

(ii) Given, $n = 102, \Sigma x = 510, \Sigma y = 7140, \Sigma x^2 = 4150, \Sigma y^2 = 740200, \Sigma xy = 54900$

We know that, regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

So, $$\bar{x} = \frac{\Sigma x}{n} = \frac{510}{102} = 5$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{7140}{102} = 70$$

$$b_{yx} = \frac{\Sigma xy - n\bar{x}\,\bar{y}}{\Sigma x^2 - n(\bar{x})^2}$$

$$= \frac{54900 - (102)(5)(70)}{4150 - 102(5)^2}$$

$$= \frac{54900 - 35700}{4150 - 2550}$$

$$= \frac{19200}{1600} = 12$$

Regression line y on x is

$$y - 70 = 12(x - 5)$$

$\Rightarrow$ $$y = 12x - 60 + 70$$

$\Rightarrow$ $$y = 12x + 10$$

$\Rightarrow$ $$y = 2(6x + 5)$$