10 CHAPTER

Vector Algebra

Level - 1

CORE SUBJECTIVE QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ) (1 Marks)

1. **Option (C) is correct.**

   Explanation: $|\vec{a}|=3, |\vec{b}|=4, |\vec{a}+\vec{b}|=5$

   We have,

   $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2(|\vec{a}|^2+|\vec{b}|^2)=2(9+16)=50$

   $\Rightarrow |\vec{a}-\vec{b}|=5.$

2. **Option (C) is correct.**

   Explanation:

   $|\vec{a}|=\sqrt{3}$

   $|b|=\dfrac{2}{\sqrt{3}}$

   $|\vec{a}\times\vec{b}|=1$ (since it is a unit vector)

   Thus,

$$\sqrt{3}\times\frac{2}{\sqrt{3}}\times\sin\theta=1$$
$$2\sin\theta=1$$
$$\sin\theta=\frac{1}{2}$$

   From trigonometry, we know that:

$$\sin\frac{\pi}{6}=\frac{1}{2}$$

   Thus, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

3. **Option (C) is correct.**

   Explanation:

$$\sin^2\theta+\cos^2\theta=1$$

   Substituting $\sin\theta=\dfrac{3}{5}$

$$\cos^2\theta=1-\left(\frac{3}{5}\right)^2$$
$$=1-\frac{9}{25}=\frac{16}{25}$$
$$\cos\theta=\pm\frac{4}{5}$$
$$\hat{a}\cdot\hat{b}=|\hat{a}||\hat{b}|\cos\theta$$

   Since unit vectors have magnitude 1, we get:

$$\hat{a}\cdot\hat{b}=1\times1\times\pm\frac{4}{5}=\pm\frac{4}{5}$$

4. **Option (D) is correct.**

   Explanation: Finding the vector with terminal point A(2, –3, 5) and initial point B(3, –4, 7):

   The vector $\overrightarrow{AB}$ is given by:

$$\overrightarrow{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

   Substituting the given points:

$$\overrightarrow{AB}=(2-3)\hat{i}+(-3+4)\hat{j}+(5-7)\hat{k}$$
$$\overrightarrow{AB}=-\hat{i}+\hat{j}-2\hat{k}$$

5. **Option (C) is correct.**

   Explanation: The dot product of two vectors is given by:

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

   Since $-1\le\cos\theta\le1$, it follows that:

$$-|\vec{a}||\vec{b}|\le\vec{a}\cdot\vec{b}\le|\vec{a}||\vec{b}|$$

   Thus, the always-true statement is:

   (C) $\vec{a}\cdot\vec{b}\le|\vec{a}||\vec{b}|$

6. **Option (A) is correct.**

   Explanation:

$$v_1=\hat{i}+\hat{k}$$
$$v_2=\hat{i}-\hat{k}$$
$$v_1\times v_2=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&1\\1&0&-1\end{vmatrix}$$
$$=\hat{i}\begin{vmatrix}0&1\\0&-1\end{vmatrix}-\hat{j}\begin{vmatrix}1&1\\1&-1\end{vmatrix}+\hat{k}\begin{vmatrix}1&0\\1&0\end{vmatrix}$$
$$=\hat{i}(0-0)-\hat{j}(-1-1)+\hat{k}(0-0)$$
$$=2\hat{j}$$

7. **Option (C) is correct.**

Explanation: Vectors are perpendicular if their dot product is zero:

$$\vec{a}\cdot\vec{b} = (2\hat{i} - \hat{j} + \hat{k})\cdot(\hat{i} + \hat{j} - \hat{k})$$
$$= (2\times 1) + (-1\times 1) + (1\times -1)$$
$$= 2 - 1 - 1 = 0$$

8. **Option (C) is correct.**

Explanation:

$|\vec{a}| = 1$, $|\vec{b}| = 2$, $\vec{a}\cdot\vec{b} = \sqrt{3}$

Let θ be the angle between $2\vec{a}$ and $-\vec{b}$:

$$\therefore \quad \cos\theta = \frac{(2\vec{a}).(-\vec{b})}{|2\vec{a}|.|-\vec{b}|} = \frac{-2\vec{a}.\vec{b}}{|2||\vec{a}||-1||\vec{b}|}$$
$$= \frac{-2.\sqrt{3}}{2\times 1\times 1\times 2} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$
$$\therefore \quad \cos\theta = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right)$$
$$= \cos\frac{5\pi}{6} \Rightarrow \theta = \frac{5\pi}{6}$$

9. **Option (B) is correct.**

Explanation:

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$$
$$\vec{b} = \hat{i} - 3\hat{j} - 3\hat{k}$$
$$\vec{c} = -3\hat{i} + 4\hat{j} + 4\hat{k}$$

To determine the nature of the triangle, we check whether the vectors satisfy the Pythagorean theorem, i.e., if:

$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{c}|^2$$

Now,

$$|\vec{a}|^2 = (2)^2 + (-1)^2 + (1)^2 = 4 + 1 + 1 = 6$$
$$|\vec{b}|^2 = (1)^2 + (-3)^2 + (-3)^2 = 1 + 9 + 9 = 19$$
$$|\vec{c}|^2 = (-3)^2 + (4)^2 + (4)^2 = 9 + 16 + 16 = 41$$

Since the sum does not match, the triangle is not right-angled.

Now, checking if it's equilateral isosceles:

$$|\vec{a}| \neq |\vec{b}| \neq |\vec{c}|$$

Since all three sides have different lengths,

$$c^2 > a^2 + b^2$$

Since 41 > 6 + 19 = 25, the triangle is obtuse-angled.

10. **Option (C) is correct.**

Explanation: We are given that $|\vec{a}| = a$

Using the formula for the magnitude of a cross product:

$$|\vec{a}\times\hat{i}| = |a||\hat{i}|\sin 90^\circ = a(1) = a$$
$$|\vec{a}\times\hat{j}| = |a||\hat{j}|\sin 90^\circ = a(1) = a$$
$$|\vec{a}\times\hat{k}| = |a||\hat{k}|\sin 90^\circ = a(1) = a$$

Now,

$$a^2 + a^2 + a^2 = 3a^2$$

11. **Option (D) is correct.**

Explanation:

$$\vec{r} = \frac{m\vec{q} + n\vec{p}}{m+n}$$

Substituting $m = 3$, $n = 1$:

$$\vec{r} = \frac{3\vec{q} + 1\vec{p}}{3+1} = \frac{\vec{p} + 3\vec{q}}{4}$$

S is the mid-point of PR, so its position vector is:

$$\vec{s} = \frac{\vec{p} + \vec{r}}{2}$$

Substituting $\vec{r} = \dfrac{\vec{p} + 3\vec{q}}{4}$

$$\vec{s} = \frac{\vec{p} + \frac{\vec{p} + 3\vec{q}}{4}}{2}$$
$$= \frac{\frac{4\vec{p} + \vec{p} + 3\vec{q}}{4}}{2}$$
$$= \frac{\frac{5\vec{p} + 3\vec{q}}{4}}{2}$$
$$= \frac{5\vec{p} + 3\vec{q}}{8}$$

12. **Option (B) is correct.**

Explanation: The projection of $\vec{a}$ on $\vec{b}$ is:

$$\text{Proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|b|}$$

Similarly, projection of $\vec{b}$ on $\vec{a}$ is:

$$\text{Proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

The required ratio is:

$$\frac{\text{Proj}_{\vec{b}}\vec{a}}{\text{Proj}_{\vec{a}}\vec{b}} = \frac{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}} = \frac{|\vec{a}|}{|\vec{b}|}$$

Now,

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + (-6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$
$$|\vec{b}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Thus, required ratio is

$$\frac{|\vec{a}|}{|b|} = \frac{7}{3}$$

13. **Option (A) is correct.**

Explanation: Given $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$

$$\Rightarrow \quad |\vec{a}|.|\vec{b}|\cos\theta = |\vec{a}|.|\vec{b}|\sin\theta$$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$ [as $0 < \theta < \pi$]

But $\vec{a}.\vec{b} > 0$

Therefore, $\theta = \dfrac{\pi}{4}$

14. Option (D) is correct.

Explanation: Since, $\vec{a}$ and $\vec{b}$ are collinear, then coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$ are proportional i.e., $\frac{2}{3} = \frac{-4}{-6}$

$$= \frac{\lambda}{1} \Rightarrow \lambda = \frac{2}{3}$$

15. Option (D) is correct.

Explanation: Since $\vec{a}$ and $\vec{b}$ are unit vectors:

$$|\vec{a} + \vec{b}| = \sqrt{|a|^2 + |b|^2 + 2(\vec{a} \cdot \vec{b})}$$

$$= \sqrt{1 + 1 + 2\cos\theta}$$

$$= \sqrt{2 + 2\cos\theta}$$

Given that $a + b$ is also a unit vector, its magnitude must be 1:

$$\sqrt{2 + 2\cos\theta} = 1$$

$$2 + 2\cos\theta = 1$$

$$2\cos\theta = -1$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}$$

16. Option (A) is correct.

Explanation:

$$v_1 = 2i + pj + k,$$

$$v_2 = -4i - 6j + 26k$$

Since the vectors are perpendicular, their dot product must be zero:

$$v_1 \cdot v_2 = 0$$

Now, $(2i + pj + k) \cdot (-4i - 6j + 26k)$

$$= (2 \times -4) + (p \times -6) + (1 \times 26)$$

$$= -8 - 6p + 26$$

$$= 18 - 6p$$

$$18 - 6p = 0$$

$$6p = 18$$

$$p = 3$$

17. Option (D) is correct.

Explanation: Compute $(\hat{i} \times \hat{j}) \cdot \hat{j} + (\hat{j} \times \hat{i}) \cdot \hat{k}$

Using vector cross product rules:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{i} = -\hat{k}$$

Thus,

$$(\hat{i} \times \hat{j}) \cdot \hat{j} = \hat{k} \cdot \hat{j} = 0$$

$$(\hat{j} \times \hat{i}) \cdot \hat{k} = (-\hat{k}) \cdot \hat{k} = -1$$

Now,

$$0 + (-1) = -1$$

18. Option (B) is correct.

Explanation:

$$\vec{b} = \hat{i} - \vec{a}$$

$$= \hat{i} - (2\hat{i} - 2\hat{j} + 2\hat{k})$$

$$= -\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\therefore \quad |\vec{b}| = \sqrt{(-1)^2 + (2)^2 + (-2)^2}$$

$$= \sqrt{1 + 4 + 4}$$

$$= \sqrt{9} = 3$$

19. Option (A) is correct.

Explanation: Direction cosines of the vector $\hat{i} - b\hat{j} + \hat{k}$ are

$$\frac{1}{\sqrt{1^2 + b^2 + 1^2}}, \frac{-b}{\sqrt{1^2 + b^2 + 1^2}}, \frac{1}{\sqrt{1^2 + b^2 + 1^2}}$$

Here, direction cosines are equal, we have

$$\frac{1}{\sqrt{2^2 + b^2}} = \frac{-b}{\sqrt{2 + b^2}} = \frac{1}{\sqrt{2 + b^2}}$$

Equating first two terms, we get

$$b = -1$$

20. Option (B) is correct.

Explanation: Let $\vec{a} = \hat{i} + \lambda\hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ on $\vec{b}$ is: $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$\Rightarrow \quad \sqrt{2} = \frac{(\hat{i} + \lambda\hat{j}).(\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}}$$

$$\Rightarrow \quad \sqrt{2} = \frac{(1 + \lambda)}{\sqrt{2}}$$

$$\Rightarrow \quad 2 = 1 + \lambda$$

$$\Rightarrow \quad \lambda = 1$$

21. Option (D) is correct.

Explanation: Given, $|\vec{a} \times \vec{b}| = \sqrt{3}$ and $\vec{a}.\vec{b} = -3$

$$\because \quad \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

$$\Rightarrow \quad |\vec{a}||\vec{b}| = \frac{\vec{a} \times \vec{b}}{\sin\theta\, \hat{n}}$$

$$\therefore \quad |\vec{a}||\vec{b}| = \frac{\sqrt{3}}{|\sin\theta||\hat{n}|}$$

Also, $\quad |\vec{a}||\vec{b}| = \frac{-3}{\cos\theta} \quad [\because \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$

Now, $\quad \frac{\sqrt{3}}{|\sin\theta|} = \frac{-3}{\cos\theta}$

$$\Rightarrow \quad \tan\theta = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = -\tan\frac{\pi}{6}$$

$$\Rightarrow \quad \tan\theta = \tan\left(\pi - \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \tan\theta = \tan\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{6}$$

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (D) is correct.**

Explanation: $\vec{b}.\vec{c}$ is a scalar because the dot product of two vectors results in a scalar.

However, multiplying this scalar by a vector a gives a vector, not a scalar.

Since $(\vec{b}.\vec{c}).\vec{a}$ is a vector, the assertion (A) is false.

The reason (R) is correct, as the dot product of two vectors is always a scalar.

Assertion (A) is false, but Reason (R) is true.

2. **Option (C) is correct.**

Explanation: The dot product is commutative, meaning

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

So the assertion (A) is true.

The cross product is anti-commutative, meaning

$$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$$

So the reason (R) is false.

3. **Option (D) is correct.**

Explanation: The projection of $\vec{a}$ on $\vec{b}$ is given by:

$$\text{Proj}_b\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

The projection of $\vec{b}$ on $\vec{a}$ is given by:

$$\text{Proj}_a\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Clearly, these two projections are not necessarily equal because the denominators are different. So, assertion (A) is false.

The reason (R) is true because the angle between two vectors remains the same regardless of which vector is projected onto the other.

Assertion (A) is false, but Reason (R) is true.

4. **Option (C) is correct.**

Explanation: The dot product formula is:

$$a\cdot b = |a|\,|b|\cos\theta$$

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

$$\vec{a}(-\vec{a}) = -\vec{a}.\vec{a}$$
$$= -|\vec{a}| = -1$$

The assertion (A) is true.

$$\cos\theta = \frac{\vec{a}.(-\vec{a})}{|\vec{a}|\,|-\vec{a}|} = \frac{-1}{1.1} = -1$$

$\therefore$ $\cos\theta = -1 = \cos\pi$

$\therefore$ $\theta = \pi$

So, the reason (R) is false because the angle is π, not $\frac{\pi}{2}$.

Assertion (A) is true, but Reason (R) is false.

5. **Option (B) is correct.**

Explanation: To check whether the vectors form a right-angled triangle, we use the Pythagorean theorem:

$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{c}|^2$$

If c is the longest side.

Magnitudes of vectors:

$$|\vec{a}|^2 = 6^2 + 2^2 + (-8)^2$$
$$= 36 + 4 + 64 = 104$$
$$|\vec{b}|^2 = 10^2 + (-2)^2 + (-6)^2$$
$$= 100 + 4 + 36 = 140$$
$$|\vec{c}|^2 = 4^2 + (-4)^2 + 2^2$$
$$= 16 + 16 + 4 = 36$$

Now,

$$104 + 36 = 140$$

Hence, it forms a right angle triangle.

So, assertion is true.

The reason (R) correctly states the condition for forming a general triangle but is not relevant to right-angled triangles. So, reason is not the explanation of assertion.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. We have $\vec{b} + \lambda\vec{c} = (-1+3\lambda)\hat{i} + (2+\lambda)\hat{j} + \hat{k}$

Since, $(\vec{b} + \lambda\vec{c}) \perp \vec{a}$, therefore

$$(\vec{b} + \lambda\vec{c}).\vec{a} = 0 \Rightarrow 2(-1+3\lambda) + 2(2+\lambda) + 3 = 0$$

$$\lambda = -\frac{5}{8}$$

2. $\vec{d}_1 = \vec{a} + \vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k},\ \vec{d}_2 = \vec{a} - \vec{b} = -6\hat{j} - 8\hat{k}$

Area of the parallelogram $= \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$$= \frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8\end{matrix}\right\|$$

$$= 2|\hat{i} + 8\hat{j} - 6\hat{k}|$$

Area of the parallelogram $= 2\sqrt{101}$ sq. units.

3. Given, $(\vec{a} + \vec{b}) \perp \vec{a}$

$\therefore$ $(\vec{a} + \vec{b}).\vec{a} = 0$

$\Rightarrow$ $|\vec{a}|^2 + \vec{b}.\vec{a} = 0$ ...(1)

Also, $(2\vec{a}+\vec{b}) \perp \vec{b}$

$$(2\vec{a}+\vec{b}).\vec{b} = 0$$

$$\Rightarrow \quad 2\vec{a}.\vec{b}+|\vec{b}|^2 = 0 \quad ...(2)$$

$$2(-|\vec{a}|^2)+|\vec{b}|^2=0 \quad \text{[Using (1) and (2)]}$$

$$|\vec{b}|^2=2|\vec{a}|^2 \Rightarrow |\vec{b}|=\sqrt{2}|\vec{a}|$$ **Hence Proved.**

4. $$\overrightarrow{DA}+\overrightarrow{DB} = \overrightarrow{AB}$$

$$\overrightarrow{DA} = (2\hat{i}-4\hat{j}+5\hat{k})-(3\hat{i}-6\hat{j}+2\hat{k})$$

$$= -\hat{i}+2\hat{j}+3\hat{k}$$

$$\overrightarrow{DA}\times\overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ 2 & -4 & 5 \end{vmatrix} = 22\hat{i}+11\hat{j}$$

$$\text{Area} = |\overrightarrow{DA}\times\overrightarrow{AB}| = |22\hat{i}+11\hat{j}|$$

$$= \sqrt{605} \text{ or } 11\sqrt{5}$$

5. Position vector of $C = \vec{r} = \dfrac{4\vec{b}-\vec{a}}{3}$

i.e. $$\vec{r} = \frac{1}{3}(-5\hat{i}+2\hat{j}+5\hat{k})$$

Now, $$\overrightarrow{AB} = -2\hat{i}-\hat{j}+2\hat{k} \Rightarrow |\overrightarrow{AB}|=3$$

$$\overrightarrow{BC} = -\frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k}) \Rightarrow |\overrightarrow{BC}|=1$$

$$|\overrightarrow{AB}|:|\overrightarrow{BC}| = 3:1$$

6. $\because$ $$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$$

As, $0 \le |\sin\theta| \le 1$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta| \le |\vec{a}||\vec{b}|$$

$$\Rightarrow |\vec{a}\times\vec{b}| \le |\vec{a}||\vec{b}|$$

For equality, $\sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2} \Rightarrow \vec{a}$ is perpendicular to $\vec{b}$.

7. Given $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}+\vec{b}-\vec{c}=0$

$\therefore$ $\vec{b}=\vec{c}-\vec{a}$

Now, $$\vec{b}.\vec{b} = (\vec{c}-\vec{a}).(\vec{c}-\vec{a})$$

$$|\vec{b}|^2 = |\vec{c}|^2-2(\vec{a}.\vec{c})+|\vec{a}|^2$$

$$1^2 = 1-2(\vec{a}.\vec{c})+1$$

$$\vec{a}.\vec{c} = \frac{1}{2}$$

Now,

$$\vec{a}.\vec{c} = |\vec{a}|.|\vec{c}|\cos\theta$$

$$\frac{1}{2} = (1).(1)\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

8. $$p = \frac{(7\hat{i}-\hat{j}+8\hat{k}).(\hat{i}+2\hat{j}+2\hat{k})}{|\hat{i}+2\hat{j}+2\hat{k}|}$$

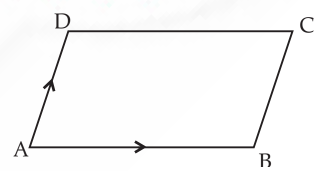
$$\left[\because \text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}\right]$$

$$= \frac{(7)(1)+(-1)(2)+(8)(2)}{\sqrt{1^2+2^2+2^2}} = 7$$

9. $$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = (2\hat{i}-4\hat{j}+5\hat{k})+(\hat{i}-2\hat{j}-3\hat{k})$$

$$= 3\hat{i}-6\hat{j}+2\hat{k}$$

Required unit vector $= \dfrac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$



10. Unit vector along $\hat{i}+\hat{j}+\hat{k}$ is $\dfrac{\hat{i}}{\sqrt{3}}+\dfrac{\hat{j}}{\sqrt{3}}+\dfrac{\hat{k}}{\sqrt{3}}$

Required vectors are $3\hat{i}+3\hat{j}+3\hat{k}$ and $-3\hat{i}-3\hat{j}-3\hat{k}$

11. According to question, $\vec{c}-\vec{a} = \dfrac{5}{4}(\vec{b}-\vec{a})$

$$\therefore \quad \vec{c} = \frac{5\vec{b}}{4}-\frac{\vec{a}}{4}$$

12. $$|\vec{a}-\vec{b}|^2 = |\vec{a}+\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2-2\vec{a}.\vec{b}+|\vec{b}|^2 = |\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2$$

$$\Rightarrow 4\vec{a}.\vec{b}=0$$

$\Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ is 90°.

13. $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \Rightarrow \vec{a}\cdot(\vec{b}-\vec{c})=0 \Rightarrow \vec{a}=\vec{0}; \vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

As, $\vec{a}\ne\vec{b}\ne\vec{0}; \vec{b}\ne\vec{c}$

$\therefore$ The angle between $\vec{a}$ and $\vec{b}-\vec{c}$ is $\frac{\pi}{2}$.

14. $\overrightarrow{BA} = 2\hat{i} - \hat{j}$; $\overrightarrow{BC} = 2\hat{k} - \hat{j}$

$|\overrightarrow{BA}| = |\overrightarrow{BC}| = \sqrt{5} \Rightarrow \Delta ABC$ is an isosceles triangle.

15. $(\vec{r}\times\hat{j}).(\vec{r}\times\hat{k}) - 12 = (3\hat{k}-6\hat{i}).(-3\hat{j}-2\hat{i}) - 12$

$= 12 - 12 = 0$

16. Here, $\left[\frac{(\hat{i}+\hat{j}+\hat{k}).(p\hat{i}+\hat{j}-2\hat{k})}{\sqrt{p^2+1+4}}\right] = \frac{1}{3}$

$\Rightarrow \quad \frac{p-1}{\sqrt{p^2+5}} = \frac{1}{3}$

$\Rightarrow \quad 8p^2 - 18p + 4 = 0$

$4p^2 - 9p + 2 = 0$

$4p^2 - 8p - p + 2 = 0$

$(4p-1)(p-2) = 0$

$\Rightarrow \quad p = 2$ or $p = \frac{1}{4}$

17. $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 2 & -2 & 1\end{vmatrix} = \hat{i} - 2\hat{j} - 6\hat{k}$

$|\vec{a}\times\vec{b}| = \sqrt{41}$

So, unit vector along $\vec{a}\times\vec{b}$ is $\frac{1}{\sqrt{41}}(\hat{i}-2\hat{j}-6\hat{k})$

18. Let θ be the angle between $\vec{a}$ and $\vec{b}$

Since $\vec{a}\times\vec{b}$ is a unit vector, we have $|\vec{a}\times\vec{b}| = 1$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 1 \Rightarrow 3\times\frac{2}{3}\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{2}$, or $\theta = 30^\circ\left(\text{or } \frac{\pi}{6}\right)$

19. Here,

$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}$

$= 20\hat{i} + 5\hat{j} - 5\hat{k}$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{400+25+25} = \sqrt{450}$

Area of parallelogram $= |\vec{a}\times\vec{b}| = \sqrt{450} = 15\sqrt{2}$ unit^2

20. Let $\vec{a} = \vec{c} + \vec{d}$, $\vec{c} \parallel \vec{b} \Rightarrow \vec{c} = \lambda\vec{b} \therefore \vec{c} = 3\lambda\hat{i} + \lambda\hat{k}$

and $\vec{d} = (5-3\lambda)\hat{i} - 2\hat{j} + (5-\lambda)\hat{k}$

$\vec{b}\cdot\vec{d} = 0 \Rightarrow 15 - 9\lambda + 5 - \lambda = 0 \Rightarrow \lambda = 2$

$\vec{a} = (6\hat{i} + 2\hat{k}) + (-\hat{i} - 2\hat{j} + 3\hat{k})$

21. $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b}-\vec{c} = 0$ or $\vec{a}\perp(\vec{b}-\vec{c})$

as $\vec{a}\neq 0 \Rightarrow \vec{b}-\vec{c} = 0$ or $\vec{a}\perp(\vec{b}-\vec{c})$ ...(1)

Again, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c} \Rightarrow \vec{a}\times(\vec{b}-\vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b}-\vec{c} = 0$ or $\vec{a}\parallel(\vec{b}-\vec{c})$

as $\vec{a}\neq 0 \Rightarrow \vec{b}-\vec{c} = 0$ or $\vec{a}\parallel(\vec{b}-\vec{c})$

From (1) and (2), $\vec{b} = \vec{c}$

($\because \vec{a}$ can't be parallel and perpendicular to $(\vec{b}-\vec{c})$ simultaneously.)

22. $\vec{a}+\vec{b}+\vec{c} = 0 \Rightarrow (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{0})^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow 49 + 576 + 625 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = -625$

23. $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & 3 & -5\end{vmatrix} = \hat{i} + 11\hat{j} + 7\hat{k}$

$\vec{a}.(\vec{a}\times\vec{b}) = 1 - 22 + 21 = 0$

$|\vec{a}||\vec{a}\times\vec{b}|\cos\theta = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

24. Let $\vec{c} = \vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$, and $\vec{d} = \vec{a}-\vec{b} = -\hat{j}-2\hat{k}$

$\vec{c}\times\vec{d} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix} = -2\hat{i} + 4\hat{j} - 2\hat{k}$

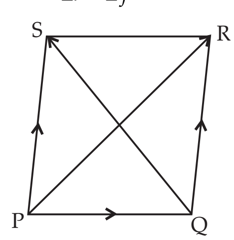
$|\vec{c}\times\vec{d}| = \sqrt{24}$

Required vector $= 6\left(\frac{\vec{c}\times\vec{d}}{|\vec{c}\times\vec{d}|}\right)$

$= \frac{6}{\sqrt{24}}(-2\hat{i}+4\hat{j}-2\hat{k})$ or $\sqrt{6}(-\hat{i}+2\hat{j}-\hat{k})$

25. $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS}$ $\quad \because \overrightarrow{PS} = \overrightarrow{QR}$

$= 2\hat{i} - 2\hat{j}$



$|\overrightarrow{PR}| = 2\sqrt{2}$

$\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PS} = \overrightarrow{PS} - \overrightarrow{PQ}$

$= -4\hat{i} + 2\hat{j} - 4\hat{k}$

$|\overrightarrow{QS}| = 6$

26. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$ $\quad$ [$\because$ Given $(\vec{a}+\vec{b})\perp r\,(\vec{a}-\vec{b})$]

$\Rightarrow \quad |\vec{a}|^2-|\vec{b}|^2 = 0$

$\Rightarrow \quad y^2+5-14=0$

$\Rightarrow \quad y = -3$ or 3

27. $|2\vec{a}+3\vec{b}| = |3\vec{a}-2\vec{b}|$

$\Rightarrow \quad |2\vec{a}+3\vec{b}|^2 = |3\vec{a}-2\vec{b}|^2$

$\Rightarrow \quad 4|\vec{a}|^2+12\vec{a}.\vec{b}+9|\vec{b}|^2 = 9|\vec{a}|^2-12\vec{a}.\vec{b}+4|\vec{b}|^2$

As $|\vec{a}|=|\vec{b}|=1$

$\therefore 24\vec{a}.\vec{b}=5|\vec{a}|^2-5|\vec{b}|^2=0\Rightarrow \vec{a}.\vec{b}=0$

So, $\vec{a}\perp\vec{b}$ or Angle between them is $\frac{\pi}{2}$

28. $\vec{b}+\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$

Projection of $\vec{b}+\vec{c}$ on $\vec{a}=\frac{(\vec{b}+\vec{c}).\vec{a}}{|\vec{a}|}$

$=\frac{(3\hat{i}+\hat{j}+2\hat{k}).(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{9}}$

$=\frac{6-2+2}{3}=\frac{6}{3}=2$

29. $|\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta=400$

$|\vec{a}|^2\,5^2(1) = 400$

$|\vec{a}|^2 = 16$

$|\vec{a}|^2 = 16$

$|\vec{a}| = 4$

30. Let the required vector be $x\hat{i}+x\hat{j}+x\hat{k}$.

$\sqrt{x^2+x^2+x^2} = 5\sqrt{3}$

$\sqrt{3}x^2 = 25\sqrt{3}$

$x^2 = 25 \Rightarrow x = \pm 5$

Required vectors are: $5\hat{i}+5\hat{j}+5\hat{k}$ or $-5\hat{i}-5\hat{j}-5\hat{k}$.

31. Let $\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{a}\cdot\vec{b}=1 \Rightarrow x+y+z=1$ ...(1)

$\vec{a}\times\vec{b} = \hat{j}-\hat{k}$

$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ x & y & z\end{vmatrix}=\hat{i}(z-y)-\hat{j}(z-x)+\hat{k}(y-x)=\hat{j}-\hat{k}$

$z-y=0\Rightarrow y=z$ ...(1)

$\Rightarrow \quad x-z=1$ ...(2)

$x-y=1$ ...(3)

Solving (1), (2), (3)

$x=1, y=0, z=0$

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. $|\vec{a}|=|\vec{b}|=|2\vec{a}+3\vec{b}|=1$

$(2\vec{a}+3\vec{b})^2=|2\vec{a}+3\vec{b}|^2$

$\Rightarrow 4|\vec{a}|^2+12\vec{a}.\vec{b}+9|\vec{b}|^2=1$

$\Rightarrow 12\ \vec{a}.\vec{b} = -12$

$\Rightarrow \vec{a}.\vec{b} = -1 \Rightarrow |\vec{a}|.|\vec{b}|\cos\theta = -1$

$\Rightarrow \cos\theta = -1$, where θ is angle between $\vec{a}$ and $\vec{b}$

Hence, $\theta = \pi$

2. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}$ be the vector perpendicular to both $\vec{a}$ & $\vec{b}$ then

$\vec{c}=\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=3\hat{j}+3\hat{k}$

Let $\vec{d}$ the vector perpendicular to both the vectors $\vec{a}$ & $\vec{b}$ and having magnitude 4,

$\therefore \vec{d}=4\cdot\frac{\vec{c}}{|\vec{c}|}=4\hat{c}=4\cdot\frac{(3\hat{i}+3\hat{j})}{\sqrt{9+9}}=2\sqrt{2}\,\hat{i}+2\sqrt{2}\,\hat{j}$

Verification:

$|\vec{d}|=\sqrt{8+8}=4; \vec{d}\cdot\vec{a}=\vec{d}\cdot\vec{b}=0\Rightarrow \vec{d}\perp\vec{a}$ and $\vec{d}\perp\vec{b}$

3. $\vec{b}+\vec{c}=2\hat{i}+3\hat{j}+2\hat{k}$

Projection of $(\vec{b}+\vec{c})$ on $\vec{a}$

$=\frac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|}=\frac{4+6+2}{\sqrt{4+4+1}}$

$= 4$

4. Since $\vec{d}\perp\vec{a}$ and $\vec{d}\perp\vec{b}\Rightarrow\vec{d}=\lambda(\vec{a}\times\vec{b})$

$\vec{d}=\lambda(\hat{i}+5\hat{j}+3\hat{k})$

$\vec{c}\cdot\vec{d}=3\Rightarrow 2\lambda+5\lambda-6\lambda=3\Rightarrow\lambda=3$

$\Rightarrow \vec{d}=3\hat{i}+15\hat{j}+9\hat{k}$

5. $\vec{a}+\vec{b}+\vec{c}=\vec{O}$

$\Rightarrow \vec{a}\times(\vec{a}+\vec{b}+\vec{c})=\vec{O}$

$\Rightarrow \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{O}$

$\Rightarrow \vec{a}\times\vec{b}=-\vec{a}\times\vec{c}, \therefore \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

Similarly, $\vec{b}\times\vec{c}=\vec{a}\times\vec{b}, \therefore \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

6. $\vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{a}-\vec{b} = -\hat{j}-2\hat{k}$

Let $\vec{C}$ be a perpendicular vector to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$.

Then $c = |(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|$

$$\therefore\ \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$\vec{c} = -2\hat{i}+4\hat{j}-2\hat{k};\ |\vec{c}| = \sqrt{24} = 2\sqrt{6}$

Required unit vector $= \dfrac{-2}{2\sqrt{6}}\hat{i} + \dfrac{4}{2\sqrt{6}}\hat{j} - \dfrac{2}{2\sqrt{6}}\hat{k}$

or $-\dfrac{1}{\sqrt{6}}\hat{i} + \dfrac{2}{\sqrt{6}}\hat{j} - \dfrac{1}{\sqrt{6}}\hat{k}$

7. $(\vec{a}+\vec{b}+\vec{c})^2 = 0$

$\vec{a}^2+\vec{b}^2+\vec{c}^2+2(\mu) = 0$

$\mu = -\dfrac{29}{2}$

8. Consider $(\vec{a}-2\vec{d})\times(2\vec{b}-\vec{c})$

$$= \vec{a}\times 2\vec{b} - \vec{a}\times\vec{c} - 4\vec{d}\times\vec{b} + 2\vec{d}\times\vec{c}$$

$$= 0$$

$\therefore (\vec{a}-2\vec{d})\,||\,(2\vec{b}-\vec{c})$

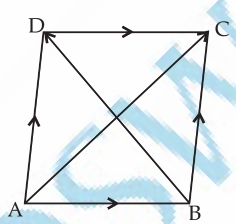
9. Let *ABCD* be a parallelogram with

$\overrightarrow{AB} = \overrightarrow{DC} = 2\hat{i}-4\hat{j}-5\hat{k}$

and $\overrightarrow{BC} = \overrightarrow{AD} = 2\hat{i}+2\hat{j}+3\hat{k}$

$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = 4\hat{i}-2\hat{j}-2\hat{k}$

and $\overrightarrow{BD} = 6\hat{j}+8\hat{k}$



$\therefore |\overrightarrow{AC}| = 2\sqrt{6}$ and $|\overrightarrow{BD}| = 10$

$\therefore$ Required unit vectors $\vec{d}_1$ and $\vec{d}_2$ are

$\vec{d}_1 = \dfrac{2}{\sqrt{6}}\hat{i} - \dfrac{1}{\sqrt{6}}\hat{j} - \dfrac{\hat{k}}{\sqrt{6}}$ and $\vec{d}_2 = \dfrac{3}{5}\hat{j} + \dfrac{4}{5}\hat{k}$

Now,

Area of || gm *ABCD* $= \dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$

$$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{matrix}\right\|$$

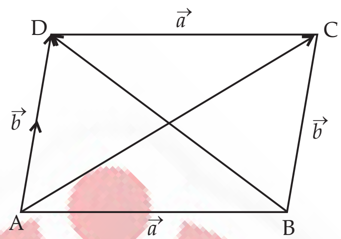
$$= \frac{1}{2}|-4\hat{i}-32\hat{j}+24\hat{k}|$$

$$= \frac{1}{2}\sqrt{1616} = 2\sqrt{101} \text{ sq. units}$$

10. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = \vec{a}+\vec{b} = \hat{i}+\hat{j}$ ...(i)

$\overrightarrow{BD} = \overrightarrow{BC}+\overrightarrow{CD} = \vec{b}-\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$ ...(ii)



On adding eqs. (i) & (ii), we get

$2\overrightarrow{AD} = 3\hat{i}+2\hat{j}+\hat{k}$

$\Rightarrow \overrightarrow{AD} = \dfrac{3}{2}\hat{i}+\hat{j}+\dfrac{1}{2}\hat{k}$

Subtracting, we get

$2\overrightarrow{AB} = -\hat{i}-\hat{k} \Rightarrow \overrightarrow{AB} = -\dfrac{1}{2}\hat{i}-\dfrac{1}{2}\hat{k}$

Area $= \dfrac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(\hat{i}-\hat{j}-\hat{k})$$

$$= \frac{\sqrt{3}}{2} \text{ sq. units}$$

11. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B, C respectively

$\vec{a} = -\hat{i}-\hat{j}+2\hat{k}$

$\vec{b} = 2\hat{i}+b\hat{j}+5\hat{k}$

$\vec{c} = 3\hat{i}+11\hat{j}+6\hat{k}$

$\overrightarrow{AB} = \vec{b}-\vec{a} = 3\hat{i}+(b+1)\hat{j}+3\hat{k}$

$\overrightarrow{AC} = \vec{c}-\vec{a} = 4\hat{i}+12\hat{j}+4\hat{k}$

As A, B, C are collinear

$$\frac{3}{4} = \frac{b+1}{12} = \frac{3}{4}$$

$\Rightarrow$ $b = 8$

$|\overrightarrow{AB}| = \sqrt{9+81+9} = \sqrt{99}$

$= 3\sqrt{11}$

$$|\vec{AC}| = \sqrt{16+144+16} = \sqrt{176} = 4\sqrt{11}$$

Here, B divides AC in the ratio 3 : 1.

12. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a}\cdot\vec{c} = 4 \Rightarrow x - y + z = 4$$

$$\vec{a}\times\vec{c} = \vec{b} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = (2\hat{i} - \hat{j} - 3\hat{k})$$

$$\Rightarrow -(y+z)\hat{i} - (z-x)\hat{j} + (y+x)\hat{k} = 2\hat{i} - \hat{j} - 3\hat{k}$$

$$\Rightarrow y+z = -2,\ z-x = 1,\ y+x = -3$$

Solving we get, $x = 0, y = -3, z = 1$

$\therefore \vec{c} = -3\hat{i} + \hat{k}$

13. One diagonal of the parallelogram

$$= (2\hat{i} - 4\hat{j} + 5\hat{k}) + (\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Unit vector parallel to the diagonal

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9+36+4}}$$

$$= \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

Vector area of parallelogram

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= |\hat{i}(22) - \hat{j}(-11) + \hat{k}(0)|$$

$$= |22\hat{i} + 11\hat{j}|$$

$\therefore$ Area $= \sqrt{484+121} = \sqrt{605} = 11\sqrt{5}$ sq. units

14. If $(\vec{a}\times\vec{b}) \perp r\ \vec{C}$, then $(\vec{a}+\lambda\vec{b}).\vec{c} = 0$

$$\Rightarrow [(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}].(3\hat{i}+\hat{j}) = 0$$

$\Rightarrow 3(2-\lambda) + (2+2\lambda).1 = 0$

$\Rightarrow -3\lambda + 2\lambda + 6 + 2 = 0$

$\Rightarrow \lambda = 8$

15. $\vec{a}+\vec{b}+\vec{c} = 0$ gives $|\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a} = 0$$

$$\Rightarrow 9 + 25 + 16 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow \vec{a}\cdot b + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -25$$

16. According to question

$$\frac{(\vec{c}+\lambda\vec{b})\cdot\vec{a}}{|\vec{a}|} = 2\sqrt{6}$$

$$\Rightarrow \frac{\vec{c}\cdot\vec{a} + \lambda\vec{b}\cdot\vec{a}}{\sqrt{6}} = 2\sqrt{6}$$

$\Rightarrow -2 + \lambda(-1) = 12$

$\Rightarrow \lambda = -14$

17. Consider $$\frac{|\vec{a}+\vec{b}|^2}{|\vec{a}-\vec{b}|^2} = \frac{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\alpha}{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\alpha}$$

$$= \frac{2m^2(1+\cos\alpha)}{2m^2(1-\cos\alpha)}$$

where $|\vec{a}| = |\vec{b}| = m$

$$= \frac{2\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}$$

$$= \cot^2\left(\frac{\alpha}{2}\right)$$

$$\therefore \frac{|\vec{a}+\vec{b}|}{|\vec{a}-\vec{b}|} = \cot\left(\frac{\alpha}{2}\right)$$ **Hence Proved.**

18. Let, $|\vec{a}| = |\vec{b}| = |\vec{c}| = m$

$$|2\vec{a}+\vec{b}+2\vec{c}|^2 = 4|\vec{a}|^2 + |\vec{b}|^2 + 4|\vec{c}|^2 + 4\vec{a}\cdot\vec{b} + 4\vec{b}\cdot\vec{c} + 8\vec{a}\cdot\vec{c}$$

$$= 9m^2 \quad [\because \vec{a}.\vec{b} = 0 = \vec{b}.\vec{c} = \vec{a}.\vec{c}]$$

$$|2\vec{a}+\vec{b}+2\vec{c}| = 3m$$

Let θ be angle between $2\vec{a}+\vec{b}+2\vec{c}$ and $\vec{a}$.

$$\cos\theta = \frac{(2\vec{a}+\vec{b}+2\vec{c})\cdot\vec{a}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{a}|}$$

$$= \frac{2m^2}{3m\cdot m} = \frac{2}{3}$$

Let, α be angle between $2\vec{a}+\vec{b}+3\vec{c}$ and $\vec{c}$.

$$\cos\theta = \frac{(2\vec{a}+\vec{b}+2\vec{c}).\vec{c}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{c}|}$$

$$= \frac{2m^2}{3m.m} = \frac{2}{3}$$

Hence, $(2\vec{a}+\vec{b}+2\vec{c})$ is equally included to both $\vec{a}$ and $\vec{c}$.

Thus, angle between $\vec{a}$ and $2\vec{a}+\vec{b}+2\vec{c}$ is $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

19.

$$|\vec{a}+\vec{b}| = |\vec{b}|$$

$$(\vec{a}+\vec{b})^2 = (\vec{b})^2$$

$$\vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b} = \vec{b}^2$$

$$\vec{a}^2 + 2\vec{a}.\vec{b} = 0$$

$$(\vec{a}+2\vec{b}).\vec{a} = 0$$

$\therefore\ (\vec{a}+2\vec{b})\perp\vec{a}$

20. Consider $|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b}).(\vec{a}-\vec{b})$

$$= |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$$

$$= 2-2\cos\theta \qquad [\because\ |\vec{a}|=|\vec{b}|=1]$$

$$= 2\left(2\sin^2\frac{\theta}{2}\right)$$

$$\therefore \qquad \sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}|$$ **Hence Proved.**

21. Area of parallelogram

$$= |(\vec{a}+3\vec{b})\times(3\vec{a}+\vec{b})|$$

$$= |(\vec{a}\times 3\vec{a})+(3\vec{b}\times 3\vec{a})+(\vec{a}\times\vec{b})+(3\vec{b}\times\vec{b})|$$

$$= |9(\vec{b}\times\vec{a})+(\vec{a}\times\vec{b})|$$

$$= 8|\vec{b}\times\vec{a}|$$

$$= 8[|\vec{b}|.|\vec{a}|\sin\theta]$$

$$= 8\times 1\times 1\times\frac{1}{2} \qquad [\because\ \theta = 30°]$$

$$= 4 \text{ sq. units}$$

22. $\vec{a}.\hat{n}=0, \vec{b}.\hat{n}=0 \Rightarrow \hat{n}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

Now, $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Here, $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix} = -2\hat{k}$

$\Rightarrow \hat{n} = -\hat{k}$

$\therefore |\vec{c}.\hat{n}| = |(\hat{i}+\hat{j}+\hat{k}).(-\hat{k})| = |-1|$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. (i) Uses the addition law of vectors to find the position vector of the point Q as:

$\vec{a}+\vec{b}$

Finds the position vector of the point U as:

$\dfrac{\vec{a}}{2}$

Assumes that V divides PU in the ratio $m : 1$ and uses the section formula to write the position vector of the point V in the ratio $m : 1$ as:

$$\overline{SV} = \frac{m\left(\frac{\vec{a}}{2}\right)+1(\vec{b})}{m+1} = \frac{m\vec{a}+2\vec{b}}{2(m+1)}$$

Assumes that V divides QS in the ratio $n : 1$ and uses the section formula to write the position vector of the point V in the ratio $n : 1$ as:

$$\overline{SV} = \frac{n(\vec{0})+1(\vec{a}+\vec{b})}{n+1} = \frac{\vec{a}+\vec{b}}{(n+1)}$$

Argues that the above two vectors represent the position vector of the same point and equates the coefficients of vectors a and b to get:

(1) $\dfrac{m}{2(m+1)} = \dfrac{1}{n+1}$ and

(2) $\dfrac{1}{m+1} = \dfrac{1}{n+1}$

Writes that $m = n$ and concludes that V divides PU and QS in the same ratio.

(ii) Substitutes $m = n$ equation (1) of the above step to get $m = n = 2$.

Concludes that the ratio is 2 : 1.

2. (i) Finds $\overline{AB} = \overline{OB} - \overline{OA} = 3\hat{i}+3\hat{j}+4\hat{k}$.

Finds $\overline{AC} = \overline{OC} - \overline{OA} = 6\hat{i}+6\hat{j}+8\hat{k}$.

Writes that $\dfrac{3}{6}=\dfrac{3}{6}=\dfrac{4}{8}=\dfrac{1}{2}$

Hence, $\overline{AC} = 2\overline{AB}$

Concludes that the starts with position vectors A, B and C are collinear.

(ii) Expresses the vector $\overline{AD}$ in terms of its components as:

$\overline{OD} - \overline{OA} = 0\hat{i}+0\hat{j}-10\hat{k}$

(iii) Finds the direction cosines of the start as:

$l = \cos 45° = \dfrac{1}{\sqrt{2}}$

$m = \cos 60° = \dfrac{1}{2}$

where, l and m denotes the direction cosines with respect to the x-axis and y-axis the z-axis, n, as:

$\dfrac{1}{2}+\dfrac{1}{4}+n^2 = 1$

$\Rightarrow n = \dfrac{1}{2}$ or $\dfrac{-1}{2}$

Uses the given information along with the above steps and finds the position vector of the required star as:

$$\overline{OP} = |\overline{OP}|(l\hat{i}+m\hat{j}+n\hat{k})$$

$$= 2\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}\pm\frac{1}{2}\hat{k}\right)$$

$$= (\sqrt{2}\hat{i}+\hat{j}\pm\hat{k})$$

Level - 2 ADVANCED COMPETENCY FOCUSED QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQs) (1 Marks)

1. **Option (B) is correct.**

   Explanation: $\theta = \tan^{-1}\left(\frac{\text{north component}}{\text{east component}}\right)$

   $= \tan^{-1}\left(\frac{400}{300}\right) = \tan^{-1}\left(\frac{4}{3}\right)$

2. **Option (D) is correct.**

   Explanation: $\theta = \tan^{-1}\left(\frac{\text{vertical}}{\text{horizontal}}\right) = \tan^{-1}\left(\frac{5}{10}\right)$

   $= \tan^{-1}(0.5)$

3. **Option (C) is correct.**

   Explanation: $\vec{F} \cdot \vec{d} = (5)(6) + (12)(8) = 30 + 96 = 126$ J

4. **Option (B) is correct.**

   Explanation: Using the determinant form of the cross product:

   Torque, $\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 4 & 3 & 0 \end{vmatrix}$

   $= \hat{i}(1 \times 0 - 0 \times 3) - \vec{j}(2 \times 0 - 0 \times 4) + \vec{k}(2 \times 3 - 1 \times 4)$

   $= 0\vec{i} - 0\vec{j} + (6-4)\vec{k} = 2\vec{k}$

   Magnitude of torque:

   $|\vec{r}| = |2\vec{k}| = 2$ Nm

5. **Option (A) is correct.**

   Explanation: Using the Pythagorean theorem to find the magnitude of the resultant velocity:

   $u = \sqrt{(6)^2 + (8)^2} = \sqrt{36+64} = \sqrt{100} = 10$ km/h

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (A) is correct.**

   Explanation: Assertion is true.

   Work done $W = \vec{F} \cdot \vec{d} = |\vec{F}|\,|\vec{d}| \cos\theta$.

   When $\theta = 90°$, $\cos 90° = 0 \Rightarrow W = 0$.

   Reason is also true. By definition of the dot product,

   $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$.

   If $\theta = 90°$, then $\cos\theta = 0 \Rightarrow \vec{A} \cdot \vec{B} = 0$

2. **Option (C) is correct.**

   Explanation: Torque is defined as:

   $\vec{\tau} = \vec{r} \times \vec{F}$

   This is a vector quantity resulting from the cross product of the position vector $\vec{r}$ and the force vector $\vec{F}$.

   Reason is false cross product gives a vector, not a scalar quantity.

3. **Option (A) is correct.**

   Explanation: Assertion is true. When a boat moves in one direction (say east) and the river flows in a perpendicular direction (say north), the actual velocity of the boat relative to the ground is the vector sum of the boat's velocity and the river's velocity.

   Reason is also true because vector addition is used to combine motions in perpendicular directions (e.g., horizontal and vertical) to find the resultant.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. $|\vec{u}+\vec{v}|^2$

   $= (\vec{u}+\vec{v}).(\vec{u}+\vec{v})$

   $= |\vec{u}|^2 + 2(\vec{u}.\vec{v}) + |\vec{v}|^2$

   $= 25 + 2(5 \times 3 \times \cos 60°) + 9$

   $= 49$

   The magnitude of the vector $(\vec{u}+\vec{v})$ as $\sqrt{49}$ or 7 units.

2. Writes that Manisha was right.

   Justifies the above answer as follows:

   $(\vec{p}\times\vec{q}) + (\vec{q}\times\vec{r}) + (\vec{r}\times\vec{p})$

   $= (5\hat{i}\times -2\hat{j}) + (-2\hat{j}\times 3\hat{k}) + (3\hat{k}\times 5\hat{i})$

   $= -10(\hat{i}\times\hat{j}) - 6(\hat{j}\times\hat{k}) + 15(\hat{k}\times\hat{i})$

   $= -6\hat{i} + 15\hat{j} - 10\hat{k}$

3. **(i)** False (F). Gives the reason that all the five vectors mentioned in (i) have different directions.

   (ii) True (T). Gives the reason that the following three vectors have the same initial point.

   $\vec{p}, \vec{u}$ and $\vec{t}$

4. Since $\overrightarrow{OC}$ is the median, C is the midpoint of $\overrightarrow{AB}$.

   Hence, writes $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

   $= \frac{1}{2}(6\hat{i} - 2\hat{j} + 10\hat{k})$

   $= 3\hat{i} - \hat{j} + 5\hat{k}$

   The length of the median as

   $|\overrightarrow{OC}| = \sqrt{(9+1+25)} = \sqrt{35}$ units.

5. Projection vector will be a zero vector.

Gives the reason that the projection of vector $\vec{u}$ on vector $\vec{v}$ is given by $\frac{\vec{u}.\vec{v}}{|\vec{v}|}$ and since the dot product is 0, the projection vector is a zero vector.

6. True.

$$\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}| \times \cos\theta,$$

where cos θ is negative when 90° < θ < 180°.

7. $\vec{p}+\vec{q}$ is represents the third side of the triangle whose other two sides are $\vec{p}$ and $\vec{q}$ with magnitude 1 unit each (as they are unit vectors).

Concludes that both $\vec{p}$ and $\vec{q}$ make equal angles with $\vec{p}+\vec{q}$ as they form an isosceles triangle.

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. Writes that $\overrightarrow{PQ}\times\overrightarrow{RS} = (\vec{b}-\vec{a})\times(-\vec{c})$.

Writes that $\overrightarrow{QR}\times\overrightarrow{PS} = (\vec{c}-\vec{b})\times(-\vec{a})$.

Writes that $\overrightarrow{RP}\times\overrightarrow{QS} = (\vec{a}-\vec{c})\times(-\vec{b})$.

Substitutes that above expression in the LHS to prove the given RHS as:

$$= (\overrightarrow{PQ}\times\overrightarrow{RS}) + (\overrightarrow{QR}\times\overrightarrow{PS}) + (\overrightarrow{RP}\times\overrightarrow{QS})$$

$$= \{(\vec{b}-\vec{a})\times(-\vec{c})\} + \{(\vec{c}-\vec{b})\times(-\vec{a})\} + \{(\vec{a}-\vec{c})\times(-\vec{b})\}$$

$$= (\vec{b}\times-\vec{c}) + (-\vec{a}\times-\vec{c}) + (\vec{c}\times-\vec{a}) + (-\vec{b}\times-\vec{a}) + (\vec{a}\times-\vec{b}) + (-\vec{c}\times-\vec{b})$$

$$= 2(\vec{c}\times\vec{b}) + 2(\vec{a}\times\vec{c}) + 2(\vec{b}\times\vec{a})$$

$$= 2(\overrightarrow{SR}\times\overrightarrow{SQ} + \overrightarrow{SP}\times\overrightarrow{SR} + \overrightarrow{SQ}\times\overrightarrow{SP})$$

= RHS

2. (i) Finds v_1, the scalar component along the x-axis as follows:

$$\cos\frac{\pi}{6} = \frac{v_1}{|\vec{v}|}$$

$$\Rightarrow \quad v_1 = \frac{\sqrt{3}}{2}\times 2 = \sqrt{3}$$

Finds v_2, the scalar component along the y-axis as follows:

$$\cos\theta = \frac{v_2}{|\vec{v}|}$$

$$\Rightarrow \quad v_2 = 2\cos\theta$$

Finds v_3, the scalar component along the z-axis as follows:

$$\cos\frac{\pi}{3} = \frac{v_3}{|\vec{v}|}$$

$$\Rightarrow \quad v_3 = \frac{1}{2}\times 2\times 1 = 1$$

Uses above three steps and $|\vec{v}| = 2$ to find θ as:

$$\sqrt{v_1^2+v_2^2+v_3^2} = 2$$

$$\Rightarrow \quad (\sqrt{3})^2 + 4\cos^2\theta + 1 = 4$$

$$\Rightarrow \quad \cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 90° \text{ or } \frac{\pi}{2}$$

(ii) Writes $\vec{v}$ in its component form as:

$$= v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$= \sqrt{3}\hat{i} + 2\cos\frac{\pi}{2}\hat{j} + \hat{k}$$

$$= \sqrt{3}\hat{i} + \hat{k}$$

CASE BASED QUESTIONS (4 Mark)

1. (i) $\overrightarrow{AV}$ = Position Vector of V – Position Vector of A

$$= -3\hat{i} + 7\hat{j} + 11\hat{k} - 7\hat{i} - 5\hat{j} - 8\hat{k}$$

$$= -10\hat{i} + 2\hat{j} + 3\hat{k}$$

Thus, $|\overrightarrow{AV}| = \sqrt{100+4+9} = \sqrt{113}$ units

(ii) $\overrightarrow{DA}$ = Position Vector of A – Position Vector of D

$$= 7\hat{i} + 5\hat{j} + 8\hat{k} - 2\hat{i} - 3\hat{j} - 4\hat{k}$$

$$= 5\hat{i} + 2\hat{j} + 4\hat{k}$$

Unit vector in the direction of $\overrightarrow{DA} = \frac{5\hat{i}+2\hat{j}+4\hat{k}}{3\sqrt{5}}$

(iii) (a) $\overrightarrow{DV} = -5\hat{i} + 4\hat{j} + 7\hat{k}$

$$\angle VDA = \cos^{-1}\left(\frac{\overrightarrow{DV}.\overrightarrow{DA}}{|\overrightarrow{DV}||\overrightarrow{DA}|}\right) = \cos^{-1}\left(\frac{11\sqrt{2}}{90}\right)$$

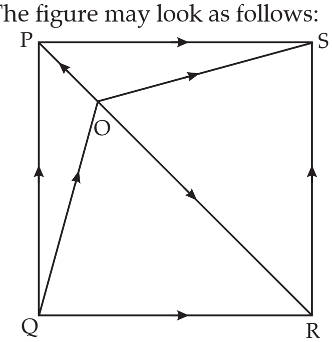
OR

(b) $\overrightarrow{DV} = -5\hat{i} + 4\hat{j} + 7\hat{k}$

$$\text{Projection of } \overrightarrow{DV} \text{ on } \overrightarrow{DA} = \left(\frac{\overrightarrow{DV}.\overrightarrow{DA}}{|\overrightarrow{DA}|}\right) = \frac{11\sqrt{5}}{15}$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. Draws the vector diagram that represents the given scenario. The figure may look as follows:



Rearranges the given equation as:

$$\overrightarrow{OS} - \overrightarrow{OP} = \overrightarrow{QO} + \overrightarrow{OR}$$

$$\Rightarrow \quad \overrightarrow{OS} + \overrightarrow{PO} = \overrightarrow{QO} + \overrightarrow{OR}$$

Uses vector addition and writes:

$$\overrightarrow{PS} = \overrightarrow{QR}$$

$\Rightarrow PS = QR$ and $PS \parallel QR$

Similarly, concludes that:

$$\overrightarrow{SR} = \overrightarrow{PQ}$$

$\Rightarrow SR = PQ$ and $SR \parallel PQ$

Uses steps 3, 4 and $\angle Q = 90°$ to write $\angle Q = \angle P = \angle R = \angle S = 90°$.

Uses steps 3, 4 and 5 to conclude that $PQRS$ is a square.

2. Considers vectors $\vec{u}$ and $\vec{v}$ as:

$$\vec{u} = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$$

Uses the condition $|\vec{u} - \vec{v}| = |\vec{u} + \vec{v}|$ to write the following:

$(p-x)^2 + (q-y)^2 + (r-z)^2 = (p+x)^2 + (q+y)^2 + (r+z)^2$

Simplifies the above equation to obtain $px + qy + rz = 0$.

Uses the above step to find the dot product of the two vectors $\vec{u}$ and $\vec{v}$ as:

$$\vec{u} \cdot \vec{v} = px + qy + rz = 0$$

Writes that the vectors are perpendicular since their dot product is equal to zero.