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CHAPTER

Continuity and Differentiability

Level - 1 CORE SUBJECTIVE QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ) (1 Mark)

1. **Option (A) is correct.**

Explanation: $xe^y = 1$

Differentiate both sides w.r.t. x

$$\frac{d}{dx}(xe^y) = \frac{d}{dx}(1)$$

$$e^y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(e^y) = 0$$

$$e^y + xe^y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{e^y}{xe^y} = -\frac{1}{x}$$

At $x = 1$:

$$\frac{dy}{dx} = -\frac{1}{1} = -1$$

2. **Option (C) is correct.**

Explanation: Given function:

$$f(x) = \sin(x^2)$$

Differentiate w.r.t. x:

$$f'(x) = \frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot\frac{d}{dx}(x^2)$$

$$= \cos(x^2).2x$$

At $x = \sqrt{\pi}$:

$$f'(\sqrt{\pi}) = \cos(\pi)\cdot 2\sqrt{\pi}$$

$$= (-1)\cdot 2\sqrt{\pi}$$

$$= -2\sqrt{\pi}$$

3. **Option (B) is correct.**

Explanation: $f(x) = [x]$ is not differentiable at integer points.

For $0 < x < 3$, the points are $x = 1, 2$.

4. **Option (C) is correct.**

Explanation: Let $u = e^{\sin^2 x}$ and $v = \cos x$

$$\therefore \quad \frac{du}{dx} = e^{\sin^2 x}(2\sin x)\cos x$$

and $$\frac{dv}{dx} = -\sin x$$

Now, $$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{e^{\sin^2 x}(2\sin x)\cos x}{-\sin x}$$

$$= -2\cos x\, e^{\sin^2 x}$$

5. **Option (C) is correct.**

Explanation: $\frac{d}{dx}[\cos x(\log + e^x)]$

$$= -\sin(\log x + e^x)\left[\left(\frac{1}{x}\right) + (e^x)\right]$$

$$= -\sin(\log x + e^x)\left(\frac{1}{x} + e^x\right)$$

Now at $x = 1$

$$\frac{d}{dx}[\cos x(\log x + e^x)] = -\sin(\log 1 + e^1)\left(\frac{1}{1} + e^1\right)$$

$$= -\sin(0 + e)(1 + e)$$

$$= -(1 + e)\sin e$$

6. **Option (D) is correct.**

Explanation: Given:

$$y = \cos^{-1}(e^x)$$

Differentiate using the chain rule:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - e^{2x}}}\cdot e^x$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{e^{-2x} - 1}}$$

7. **Option (D) is correct.**

Explanation: $f(x) = |1 - x + |x||$

Case 1: $x \geq 0$

$|x| = x$, so

$f(x) = |1 - x + x| = |1| = 1$

Case 2: $x < 0$

$$|x| = -x \text{ so}$$
$$f(x) = |1 - x - x| = |1 - 2x|$$

Since $1 - 2x > 0$ for $x < 0$,

$$f(x) = 1 - 2x$$

At $x = 0: \lim_{x\to 0^-} f(x) = 1, \lim_{x\to 0^+} f(x) = 1$, so continuous.

At $x = 1: \lim_{x\to 1^-} f(x) = 1, \lim_{x\to 1^+} f(x) = 1$, so continuous.

8. Option (C) is correct.

Explanation: Required derivative is given as:

$$\frac{d}{dx}\left(2^x\right) \div \frac{d}{dx}\left(3^x\right)$$

Using the derivative formula $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2, \frac{d}{dx}\left(3^x\right) = 3^x \ln 3$$

$$\therefore \quad \frac{d}{dx}(2^x) \div \frac{d}{dx}(3^x) = \left(\frac{2}{3}\right)^x \cdot \frac{\ln 2}{\ln 3}$$

9. Option (C) is correct.

Explanation: Differentiate 5^x and e^x using the derivative formula:

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

we get:

$$\frac{d}{dx}(5^x) = 5^x \ln 5$$
$$\frac{d}{dx}(e^x) = e^x$$

Now,

$$\frac{\frac{d}{dx}\left(5^x\right)}{\frac{d}{dx}\left(e^x\right)} = \frac{5^x \ln 5}{e^x}$$

$$= \left(\frac{5}{e}\right)^x \cdot \ln 5$$

10. Option (C) is correct.

Explanation: Given:

$$y = \sin^{-1} x$$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Again differential ting w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{x}{(1-x^2)^{3/2}}$$

Using $\sec y = \frac{1}{\sqrt{1-x^2}}$ and $\tan y = \frac{x}{\sqrt{1-x^2}}$, we get:

$$\frac{d^2y}{dx^2} = \sec^2 y \tan y$$

Alter Method

$$x = \sin y$$

$$1 = \cos y \, \frac{dy}{dx}$$

$$\sec y = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \sec y \tan y \, \frac{dy}{dx} = \sec^2 y \tan y$$

11. Option (B) is correct.

Explanation: Differentiate

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{\frac{d}{dx}\left(e^{2x}\right)}{\frac{d}{dx}\left(e^x\right)} = \frac{2e^{2x}}{e^x}$$

$$= 2e^x$$

12. Option (B) is correct.

Explanation: For continuity at $x = 0$, we need:

$$\lim_{x\to 0} f(x) = f(0) = k$$

Given:

$$f(x) = \frac{\sqrt{4+x} - 2}{x}, \quad x \neq 0$$

$$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{2} = \lim_{x\to 0} \frac{(\sqrt{4+x} - 2)(\sqrt{4+x} + 2)}{x(\sqrt{4+x} + 2)}$$

(On rationalising)

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{4+x} + 2)}$$

$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

For continuity, $k = \frac{1}{4}$

13. Option (B) is correct.

Explanation: Given, $e^{x^2 y} = c$

On differentiating w.r.t. x, both sides, we get

$$e^{x^2 y}\left(x^2 \frac{dy}{dx} + 2xy\right) = 0$$

Since $e^{x^2 y} \neq 0$,

$$x^2 \frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = \frac{-2y}{x}$$

14. Option (A) is correct.

Explanation: For continuity at $x = 4$:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$

Solve:

$$16 - c^2 = 4c + 20$$
$$c^2 + 4c + 4 = 0$$
$$(c + 2)^2 = 0 \Rightarrow c = -2$$

15. **Option (A) is correct.**

Explanation: We need to find:

$$\frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(x^3)}$$

Differentiate

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \quad \frac{\frac{d}{dx}(x^2)}{\frac{d}{dz}(x^3)} = \frac{2x}{3x^2} = \frac{2}{3x}$$

16. **Option (A) is correct.**

Explanation: The given function is:

$$f(x) = |x| + |x-2|$$

Absolute value functions are continuous everywhere, so $f(x)$ is continuous for all x, including $x = 0$ and $x = 2$.

For Differentiability

A function is not differentiable where it has a corner, which happens at points where the absolute value terms switch expressions.

At $x = 0$:

$|x|$ has a sharp corner at $x = 0$, so $f(x)$ is not differentiable at $x = 0$.

At $x = 2$:

$|x - 2|$ has a sharp corner at $x = 2$, so $f(x)$ is not differentiable at $x = 2$.

17. **Option (C) is correct.**

Explanation: For continuity at $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Left-hand limit ($x < 0$):

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+2) = 0+2 = 2$$

Right-hand limit ($0 \le x \le 1$):

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x = e^0 = 1$$

Function value:

$$f(0) = e^0 = 1$$

Since $\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x), f(x)$ is discontinuous at $x = 0$.

For continuity at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

Left-hand limit ($0 \le x \le 1$):

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^x = e^1 = e$$

Right-hand limit ($x > 1$):

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$$

Function value:

$$f(1) = e^1 = e$$

Since $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x), f(x)$ is discontinuous at $x = 1$.

So, $f(x)$ is discontinuous at $x = 0$ at $x = 1$.

18. **Option (C) is correct.**

Explanation: Given $y = f\left(\frac{1}{x}\right)$ and $f'(x) = x^3$,

Since $f'\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$, we get:

$$\frac{dy}{dx} = f'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{1}{x^3}\cdot\left(-\frac{1}{x^2}\right) = -\frac{1}{x^5}$$

At $x = \frac{1}{2}$:

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{1}{2}\right)^5} = -32$$

19. **Option (A) is correct.**

Explanation:

$$y = \log\left(\sec^{1/2}(\sqrt{x})\right) = \frac{1}{2}\log(\sec\sqrt{x})$$

Differentiate y w.r.t. 'x',

Using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sec\sqrt{x}}\cdot\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\tan\sqrt{x}}{4\sqrt{x}}$$

at $x = \frac{\pi^2}{16}$

$$\sqrt{x} = \frac{\pi}{4}, \quad \tan\frac{\pi}{4} = 1$$

$$\frac{dy}{dx} = \frac{1}{4\times(\pi/4)} = \frac{1}{\pi}$$

20. **Option (B) is correct.**

Explanation: Given:

$$x = 3\cos\theta, \quad y = 5\sin\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta, \quad \frac{dy}{d\theta} = 5\cos\theta$$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos\theta}{-3\sin\theta} = -\frac{5}{3}\cot\theta$$

21. **Option (B) is correct.**

Explanation: We are given the parametric equations:

$$x = at^2, \; y = 2at$$

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

Using the chain rule:

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

22. Option (B) is correct.

Explanation: Given:

$$y = \sec(\tan^{-1} x)$$

Let $\theta = \tan^{-1}x$, so $\tan\theta = x$ and

$$\sec\theta = \sqrt{1+x^2}$$

$$y = \sqrt{1+x^2}$$

Differentiate:

$$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

At $x = 1$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

23. Option (D) is correct.

Explanation: For continuity at $x = \frac{\pi}{2}$;

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

Given:

$$f(x) = \frac{k\cos x}{\pi - 2x}, \quad x \neq \frac{\pi}{2}$$

Let $x = \frac{\pi}{2} - h$, where $h \to 0$. Then:

$$\cos x = \cos\left(\frac{\pi}{2} - h\right) = \sin h$$

Also,

$$\pi - 2x = \pi - 2\left(\frac{\pi}{2} - h\right) = 2h$$

Thus, at $x = \frac{\pi}{2} - h$,

$$\lim_{x\to\frac{\pi^{-2}}{2}} = \frac{k\cos x}{\pi - 2x} = \lim_{x\to 0}\frac{k\sin h}{2h}$$

Using the limit property $\frac{\sin h}{h} \to 1$ as $h \to 0$:

$$\lim_{h\to 0}\frac{k\sin h}{2h} = \frac{k}{2}$$

Limit Equal to $f\left(\frac{\pi}{2}\right)$

$$\frac{k}{2} = 5$$

$$k = 10$$

24. Option (D) is correct.

Explanation: For continuity at $x = 2$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

Right-hand limit:

$$\lim_{x\to 2^+} (3x+5) = 3(2)+5 = 11$$

Left-hand limit:

$$\lim_{x\to 2^-} (kx^2) = k(2)^2 = 4k$$

Equating limits:

$$4k = 11$$

$$k = \frac{11}{4}$$

25. Option (D) is correct.

Explanation: For continuity at $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Right-hand limit:

$$\lim_{x\to 0^+} \cos x = 1$$

Left-hand limit:

$$\lim_{x\to 0^-} k\left(3x^2 - 5x\right) = k(0) = 0$$

Since $0 \neq 1$, no k satisfies the condition.

26. Option (A) is correct.

Explanation: For continuity at $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

$f(x)$ is continuous.

For differentiability:

$$f'(x) = 2x \text{ for } x > 0,\ f'(x) = -2x \text{ for } x < 0$$

$$f'(0^+) = 0,\ f'(0^-) = 0$$

Since $f'(0^+) = f'(0^-)$, $f(x)$ is differentiable.

27. Option (B) is correct.

Explanation: Given:

$$\tan\left(\frac{x+y}{x-y}\right) = k$$

Differentiate both sides w.r.t. x using implicit differentiation:

$$\sec^2\left(\frac{x+y}{x-y}\right)\cdot\frac{(x-y)\left(1+\frac{dy}{dx}\right)-(x+y)\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$(x-y)\left(1+\frac{dy}{dx}\right)-(x+y)\left(1-\frac{dy}{dx}\right) = 0$$

$$\left[\text{Since, } \sec^2\left(\frac{x+y}{x-y}\right) \neq 0\right]$$

$$x - y + (x-y)\frac{dy}{dx} - x - y + (x+y)\frac{dy}{dx} = 0$$

$$(x-y)\frac{dy}{dx} + (x+y)\frac{dy}{dx} = 2y$$

$$(x-y+x+y)\frac{dy}{dx} = 2y$$

$$\frac{dy}{dx} = \frac{y}{x}$$

28. Option (C) is correct.

Explanation: Let $y = x^{2x}$

Taking log both sides, we get

$$\log y = 2x\log x$$

Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = 2\left(1.\log x + x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = 2y\,(1 + \log x)$$

$$\frac{dy}{dx} = 2x^{2x}\,(1 + \log x)$$

29. Option (D) is correct.

Explanation: Given, $x = A\cos 4t + B\sin 4t$

$$\therefore \quad \frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t$$

and $$\frac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t$$

or, $$\frac{d^2x}{dt^2} = -16(A\cos 4t + B\sin 4t)$$

$$\frac{d^2x}{dt^2} = -16x$$

30. Option (B) is correct.

Explanation: The greatest integer function is continuous only for non-integral points i.e., not an integers.

ASSERTION-REASON QUESTIONS (1 Mark)

1. Option (A) is correct.

***Explanation:* Assertion:** We have, $f(x) = |x| + |x-1|$

Here, we have 2 critical points $x = 0$ and $x - 1 = 0$

i.e., $x = 0$ and $x = 1$

The graph of $f(x)$ has two sharp points at $x = 0$ and $x = 1$, hence $f(x)$ is not differentiable at $x = 0$ and $x = 1$.

Reason: Here reason describes the differentiability test

i.e., $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{-h}$ is left hand derivative at $x = c$

and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ is right hand derivative at $x = c$.

Since function is not differentiable at $x = c$, LHD $\neq$ RHD reason is true.

Since, derivative test can be used to check differentiability at $x = 0$ and $x = 1$.

Thus, we used the concept mentioned in reason to check assertion.

Therefore, reason is a correct explanation for assertion.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. $y = \tan^{-1}x$ and $z = \log_e x$

Then $$\frac{dy}{dx} = \frac{1}{1+x^2}$$

and $$\frac{dz}{dx} = \frac{1}{x}$$

So, $$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$= \frac{\frac{1}{1+x^2}}{\frac{1}{x}} = \frac{x}{1+x^2}$$

2. Let $y = (\cos x)^x$. Then, $y = e^{x\log_e \cos x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x\log_e \cos x}\frac{d}{dx}(x\log_e \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x\left\{\log_e \cos x\frac{d}{dx}(x) + x\frac{d}{dx}(\log_e \cos x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x\left\{\log_e \cos x + x\cdot\frac{1}{\cos x}(-\sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x\,(\log_e \cos x - x\tan x).$$

3. $$f(x) = \begin{cases} x^3, & x \ge 0 \\ -x^3, & x < 0 \end{cases}$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} h^2 = 0$$

$$\text{LHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{-h} = \lim_{h\to 0}(-h^2) = 0$$

$\because$ RHD = LHD = 0.

So $f(x)$ is differentiable at $x = 0$.

4. $$y = \sqrt{\tan\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{2\sqrt{\tan\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$

$$\sqrt{x}\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{4\sqrt{\tan\sqrt{x}}}$$

$$= \frac{1+(\tan\sqrt{x})^2}{4\sqrt{\tan\sqrt{x}}} = \frac{1+y^4}{4y}$$

5. We have, $y^2 = \cos x + y$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\Rightarrow \quad (2y-1)\frac{dy}{dx} = -\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin x}{1-2y}$$

6. $$f(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$$

At $x = 0$

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\left(\frac{h^3}{-h}\right) = \lim_{h\to 0}(-h^2) = 0$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\left(\frac{h^3}{-h}\right) = \lim_{h\to 0}(h^2) = 0$$

$\because$ LHD = RHD at $x = 0$; when $x \neq 0$. $f(x)$ is a polynomial and hence differentiable.

$\therefore$ $f(x)$ is differentiable at all points.

7. $f(x) = [x]$ at $x = -3$

$$\text{RHD} = \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h}$$

$$= \lim_{h\to 0} \frac{-3-(-3)}{h} = 0$$

$$\text{LHD} = \lim_{h\to 0} \frac{f(-3-h)-f(-3)}{-h}$$

$$= \lim_{h\to 0} \frac{-4-(-3)}{h} = \lim_{h\to 0}\left(\frac{-1}{h}\right)$$

= not defined

$\because$ LHD $\neq$ RHD

So f is not differentiable at $x = -3$.

8. $$\frac{1}{3}x^{\frac{-2}{3}} + \frac{1}{3}y^{\frac{-2}{3}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^{\frac{-2}{3}}}{y^{\frac{-2}{3}}} = \frac{-y^{2/3}}{x^{2/3}}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{8},\frac{1}{8}\right)} = \frac{-4}{4} = -1$$

9. $$f(x) = -\tan 2x, \frac{\pi}{4} < x < \frac{\pi}{2}$$

$$f'(x) = -2\sec^2 2x, \frac{\pi}{4} < x < \frac{\pi}{2}$$

$$f'\left(\frac{\pi}{3}\right) = -2\sec^2 2\left(\frac{\pi}{3}\right)$$

$$= -2\left[\sec\left(\pi - \frac{\pi}{3}\right)\right]^2$$

$$= -2\sec^2\frac{\pi}{3}$$

$$= -2(-2)^2 = -8$$

10. $$y = \sqrt{1+\cot^2(\cot^{-1}x)} = \sqrt{1+x^2}$$

$[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sqrt{1+x^2}\frac{dy}{dx} - x = 0$$

11. $$f(x) = |\cos x| = \begin{cases} \cos x & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

$$\text{LHD at } \frac{\pi}{2} = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}-h\right) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{\sin h}{-h} = -1$$

$$\text{RHD at } \frac{\pi}{2} = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-\cos\left(\frac{\pi}{2}+h\right) - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} = 1$$

LHD $\neq$ RHD

$\therefore$ f is not differentiable at $x = \frac{\pi}{2}$

12. $$\frac{dy}{dx} = 2A\cos 2x - 2B\sin 2x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -4A\sin 2x - 4B\cos 2x = -4y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 4y = 0$$

$$\Rightarrow \quad k = -4$$

13. $$x = e^{\frac{x}{y}} \Rightarrow \log x = \frac{x}{y} \Rightarrow y = \frac{x}{\log x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\log x)(1) - x\left(\frac{1}{x}\right)}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$$

14. LHD at $x = 1$

$$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{[(1-h)^2+1]-2}{-h} = 2$$

RHD at $x = 1$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[3-(1+h)]-2}{h} = -1$$

as LHD ≠ RHD, so $f(x)$ is not differentiable at $x = 1$.

15. $y = \cos^3(\sec^2 2t)$

$\Rightarrow \frac{dy}{dt} = -3\cos^2(\sec^2 2t).\sin(\sec^2 2t)$

$\times 2 \sec 2t.\sec 2t \tan 2t.2$

$\therefore \frac{dy}{dt} = -12 \cos^2 (\sec^2 2t) \times \sin(\sec^2 2t) \times \sec^2 2t$

$\times \tan 2t$

16. As, $x^y = e^{x-y} \Rightarrow \log(x^y) = \log(e^{x-y})$

$\Rightarrow \quad y \log x = (x-y) \Rightarrow y = \frac{x}{1+\log x}$

Now, differentiating both the sides wrt x

$$\frac{dy}{dx} = \frac{(\log x+1).1 - x\left(\frac{1}{x}\right)}{(\log x+1)^2} = \frac{\log x}{(1+\log x)^2}$$

17. $\lim_{x\to 0} f(x) = \lim_{x\to 0} x.\sin\frac{1}{x} = 0 \times$ Finite value in $[-1, 1] = 0$

$= f(0)$

$\therefore f$ is continuous function.

18. LHD $= \lim_{x\to 5^-} \frac{|x-5|-0}{x-5} = \lim_{x\to 5^-} \frac{-(x-5)}{x-5} = -1$

RHD $= \lim_{x\to 5^-} \frac{|x-5|-0}{x-5} = \lim_{x\to 5^-} \frac{(x-5)}{x-5} = 1$

LHD ≠ RHD, $\therefore f$ is not differentiable at $x = 5$

19. $$\frac{dx}{dy} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$

Differentiating again with respect to 'x', we get

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

20. $y^x = x^y \Rightarrow x \log y = y \log x$

Differentiating with respect to 'x',

$$\frac{x}{y}\frac{dy}{dx} + \log y = \frac{y}{x} + \frac{dy}{dx}\log x$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$

21. $$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$\therefore \frac{dx}{dy} = \frac{\sin^2(a+y)}{\sin a}$ Hence Proved

22. Given $y = (\cos x)^x + \cos^{-1}\sqrt{x}$

Let $y = u + v$

where, $u = (\cos x)^x$ and $v = \cos^{-1}\sqrt{x}$

Now, $u = (\cos x)^x$

Taking by both sides, we get

$\log u = x \log (\cos x)$

$\therefore \frac{1}{u}\frac{du}{dx} = x.\frac{1}{\cos x}(-\sin x) + 1.\log(\cos x)$

$\frac{du}{dx} = u[(-x\tan x) + \log(\cos x)]$

$\frac{du}{dx} = (\cos x)^x [\log (\cos x) - x \tan x]$

and $v = \cos^{-1}\sqrt{x}$

$\therefore \frac{dv}{dx} = \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}}$

$\frac{dv}{dx} = \frac{-1}{2\sqrt{x}\sqrt{1-x}} = \frac{1-}{2\sqrt{x-x^2}}$

Therefore,

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$\frac{dy}{dx} = (\cos x)^x [\log (\cos x) - x \tan x] \; 2\sqrt{x-x^2}$

23. Given, $y = e^{a\cos^{-1}x}$ place x in front of cos

$\therefore \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right)$

$\sqrt{1-x^2}\left(\frac{dy}{dx}\right) = -a\, e^{a\cos^{-1}x}$

$\Rightarrow \sqrt{1-x^2}\left(\frac{dy}{dx}\right) = -ay$

Squaring both sides

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating again w.r.t. x,

$$(1-x^2)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right) = a^2 y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right) - a^2 y = 0$$ **Hence proved.**

24. Let $y = u - v$

Now, $u = x^{\cos x}$

Taking log both sides, we get

$\log u = \cos x \log x$

Differentiating both sides, we get

$$\frac{1}{u}\frac{du}{dx} = -\sin x \log x + \frac{\cos x}{x}$$

$$\frac{du}{dx} = u\left(-\sin x \log x + \frac{\cos x}{x}\right)$$

$$\frac{du}{dx} = x^{\cos x}\left(-\sin x \log x + \frac{\cos x}{x}\right)$$

and $v = 2^{\sin x}$

$$\frac{dv}{dx} = 2^{\sin x}\cos x \log 2$$

(Differentiating both sides)

Therefore,

$$y = x^{\cos x}\left(-\sin x \log x + \frac{\cos x}{x}\right) - 2^{\sin x}_{\cos x \log 2}$$

25. $$x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y,

$$\frac{dx}{dy} = \frac{\cos(a+y).(-\sin y) - \cos y.[-\sin(a+y)]}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)} \Rightarrow \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

26. $$\frac{dy}{dx} = 2\left(x+\sqrt{x^2-1}\right)\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$$\frac{dy}{dx} = \frac{2(x+\sqrt{x^2-1})^2}{\sqrt{x^2-1}}$$

$$\sqrt{x^2-1}\,\frac{dy}{dx} = 2y$$

$$(x^2-1)\left(\frac{dy}{dx}\right)^2 = 4y^2$$ **Hence proved.**

27. Given, $x = \sqrt{a^{\tan^{-1}t}}$, $y = \sqrt{a^{\cot^{-1}t}}$

or $x = a^{\frac{1}{2}\tan^{-1}t}$, $y = a^{\frac{1}{2}\cot^{-1}t}$

$$\therefore \frac{dx}{dt} = a^{\frac{1}{2}\tan^{-1}t}\log a \frac{1}{2}\frac{1}{1+t^2}$$

and $$\frac{dy}{dt} = a^{\frac{1}{2}\cot^{-1}}\log a \frac{1}{2}\left(\frac{-1}{1+t^2}\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2(1+t^2)} \text{ and } \frac{dy}{dt} = \frac{-y\log a}{2(1+t^2)}$$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}$$

$$\therefore x\frac{dy}{dx} + y = 0$$ **Hence proved.**

28. $y = \sqrt{ax+b} \Rightarrow y^2 = ax + b$

Differentiate with respect to 'x', $2y\frac{dy}{dx} = a$

Differentiate again with respect to 'x',

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$$

29. $f(x)$ is differentiable in (0, 2)

$\Rightarrow f(x)$ is continuous at (0, 2)

$\Rightarrow f(x)$ is continuous at $x = 1$.

$$\therefore \lim_{x\to 1^-}(ax+b) = \lim_{x\to 1^-}(2x^2 - x) \Rightarrow a + b = 1$$

Also, $f(x)$ is differentiable at $x = 1$,

$\therefore$ L.H.D. ($x = 1$) = R.H.D. ($x = 1$)

$\Rightarrow a = 4x - 1$

$\Rightarrow a = 4(1) - 1 \therefore a = 3$ & $b = 1 - a = -2$

30. $$y = x^{1/x}$$

$$\Rightarrow \log y = \frac{1}{x}\log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{\log x}{x^2} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}}\frac{(1-\log x)}{x^2} = \frac{1(1-0)}{1}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 1$$

31. $$\frac{dx}{dt} = 2a\cos 2t$$

$$\frac{dy}{dt} = 2a\left(-\sin 2t + \frac{\sec^2 t}{2\tan t}\right)$$

$$= 2a\frac{\cos^2 2t}{\sin 2t}$$

$$\frac{dy}{dx} = \cot 2t$$

32. Given $xy = e^{x-y}$, gives $x - y = \log x + \log y$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{1}{y}+1\right)\frac{dy}{dx} = 1-\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x-1}{x}\times\frac{y}{1+y} = \frac{y(x-1)}{x(y+1)}$$

Hence proved.

33. $(x^2 + y^2)^2 = xy$ gives

$$2(x^2+y^2)\left[2x+2y\frac{dy}{dx}\right] = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad [4y(x^2+y^2)-x]\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$$

34. Here,

$$\text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = 2$$

$$\text{LHD} = \lim_{h\to 0}\left[\frac{f(1-h)-f(1)}{-h}\right] = 1$$

Since RHD $\neq$ LHD

$\therefore$ f is not differentiable at $x = 1$.

35.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin^2\lambda x}{x^2}\right)$$

$$= \lim_{x\to 0}\left[\frac{\sin^2\lambda x}{(\lambda x)^2}\cdot\lambda^2\right] = \lambda^2$$

Since $f(x)$ is continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

36. Here,

$$f(x) = \frac{1-\cos x}{2x^2} = \frac{2\sin^2\frac{x}{2}}{2x^2} = \left(\frac{\sin\frac{x}{2}}{2\frac{x}{2}}\right)^2$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{4}$$

So, if f s continuous at $x = 0$, then $f(0) = \lim_{x\to 0} f(x)$

$$\Rightarrow k = \frac{1}{4}$$

37. Given $x = a\cos t$ and $y = b\sin t$, we have

$$\frac{dx}{dt} = -a\sin t$$

and $$\frac{dy}{dt} = b\cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dt/dx}\cdot$$

$$= \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\cdot\frac{dt}{dx}$$

$$= \frac{b}{a}\ \text{cosec}^2 t.\frac{1}{-a\sin t}$$

$$= -\frac{b}{a^2}\cdot\frac{1}{\sin^3 t} \text{ or } -\frac{b}{a^2}\text{cosec}^3 t$$

38. $y = x^x \Rightarrow \log y = x\log x$, differentiating with respect to 'x', we get

$\frac{dy}{dx} = y(1+\log x)$, differentiating with respect to 'x', we get

$$\frac{d^2y}{dx^2} = \frac{y}{x}+(1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x} = 0$$

Hence proved.

39. As f is continuous at $x = 2$

$$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\lim_{x\to 2^+} 3x = \lim_{x\to 2^-}(2x+2) = k$$

$\Rightarrow k = 6$

40. $$f(x) = \frac{4+x^2}{x(2-x)(2+x)}$$

Clearly f is not continuous when $x(2 - x)(2 + x) = 0$

$\Rightarrow x = 0, 2, -2$

41. As f is continuous at $x = 1 \Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$$\lim_{x\to 1^+}(3ax+b) = \lim_{x\to 1^-}(5ax-2b) = 11$$

$\Rightarrow 3a + b = 11$ and $5a - 2b = 11$

Solving, we get $a = 3, b = 2$

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. $$\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$$

Put $x = \sin\theta$, $y = \sin\phi$

$$\Rightarrow \quad \cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = 2a\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad \cot\left(\frac{\theta-\phi}{2}\right) = a$$

$$\Rightarrow \quad \theta - \phi = 2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$

On differentiating both sides, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$ **Hence proved.**

2. $y = (\tan x)^x$

$\log y = x \log(\tan x)$ [Taking log both sides]

On differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{\sec^2 x}{\tan x}\right) + \log(\tan x)$$

$$\frac{dy}{dx} = (\tan x)^x\left[\left(\frac{x\sec^2 x}{\tan x}\right) + \log(\tan x)\right]$$

3. $$\frac{dx}{dt} = e^{\cos 3t} \times (-\sin 3t) \times 3$$

$$\frac{dy}{dt} = e^{\sin 3t} \times (\cos 3t) \times 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^{\sin 3t} \times (\cos 3t)}{-e^{\cos 3t} \times (\sin 3t)}$$

Also, $x = e^{\cos 3t} \Rightarrow \cos 3t = \log x$

$y = e^{\sin 3t} \Rightarrow \sin 3t = \log y$

$$\therefore \quad \frac{dy}{dx} = \frac{-y\log x}{x \log y}$$ **Hence proved.**

4. $$\frac{d(|x|)}{dx} = \frac{d(\sqrt{x^2})}{dx}, x \neq 0$$

$$= \frac{1}{2}(x^2)^{-\frac{1}{2}} \times \frac{d(x^2)}{dx}$$

$$= \frac{1}{2\sqrt{x^2}}2x = \frac{x}{|x|}$$ **Hence proved.**

5. We have, $x^{30}y^{20} = (x+y)^{50}$

Taking log of both sides, we get

$30\log x + 20\log y = 50\log(x+y)$

Differentiating both sides w.r.t. x, we get

$$\frac{30}{x} + \frac{20}{y}\frac{dy}{dx} = \frac{50}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{20x-30y}{y(x+y)}\right) = \frac{20x-30y}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ **Hence proved.**

6. We have, $5x + 5y = 5^{x+y}$

Differentiating both sides w.r.t. x, we get

$$5^x\log 5 + 5^y \log 5\frac{dy}{dx} = 5^{x+y}\log 5\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 5^x + 5^y\frac{dy}{dx} = 5^{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}(5^y - 5^{x+y}) = 5^{x+y} - 5^x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{5^{x+y}-5^x}{5^y - 5^{x+y}} = \frac{5^x(5^y-1)}{5^y(1-5^x)}$$

7. Differentiating both sides w.r.t. x, we get

$$y' = \frac{2\log x}{x}$$

$$\Rightarrow \quad xy' = 2\log x$$

$$\Rightarrow \quad xy'' + y' = \frac{2}{x}$$

$$\Rightarrow \quad x^2y'' + xy' = 2$$

8. $$\frac{dy}{dx} = \cos(\tan^{-1}(e^x)) \times \frac{e^x}{1+e^{2x}}$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \cos\frac{\pi}{4} \times \frac{1}{2} = \frac{1}{2\sqrt{2}}$$

9. $x\cos(p+y) + \cos p \sin(p+y) = 0$

$$\Rightarrow \quad x = \frac{-\cos p \sin(p+y)}{\cos(p+y)}$$

$$\Rightarrow \quad x = -\cos p.\tan(p+y)$$

$$\Rightarrow \quad \frac{dx}{dy} = -\cos p.\sec^2(p+y)$$

$$\Rightarrow \quad \cos p\frac{dy}{dx} = -\cos^2(p+y)$$ **Hence proved.**

10. $$f(x) = \begin{cases} \frac{x-2}{-(x-2)} + a & ; x < 2 \\ a + b & ; x = 2 \\ \frac{x-2}{(x-2)} + b & ; x > 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -1 + a & ; x < 2 \\ a + b & ; x = 2 \\ 1 + b & ; x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = -1 + a$$

$$\lim_{x\to 2^+} f(x) = 1 + b$$

and

$$f(2) = a + b$$

As f is continuous at $x = 2$

$\therefore -1 + a = 1 + b = a + b$

$\Rightarrow a = 1, b = -1$

11. As, $y = (\sin x)^x . x^{\sin x} + a^x = u + a^x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{d(a^x)}{dx}$$

where $u = (\sin x)^x . x^{\sin x} \Rightarrow \log u = x \log(\sin x) + \sin x . \log x$

On differentiating, u both sides with respect to x, we get

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x . x^{\sin x}$$

$$\left[\log(\sin x) + x \cot x + \frac{\sin x}{x} + \log x . \cos x\right]$$

Thus, $\frac{dy}{dx} = (\sin x)^x . x^{\sin x} \times$

$$\left[\log(\sin x) + x \cot x + \frac{\sin x}{x} + \log x . \cos x\right] + a^x \log a$$

12. Taking 'log' on both sides of $(\cos x)^y = (\cos y)^x$, we get

$$y \log \cos x = x \log \cos y$$

$$\Rightarrow \frac{dy}{dx} \log \cos x + y(-\tan x) = \log \cos y + x(-\tan y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log \cos y + y \tan x}{\log \cos x + x \tan y}$$

13. $y = (\tan^{-1} x)^2$

Differentiating w.r.t. x, both sides, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2 \tan^{-1} x}{1 + x^2}$$

$\Rightarrow (1 + x^2)\frac{dy}{dx} = 2 \tan^{-1} x$, differentiating again with respect to 'x', we get

$$2x\left(\frac{dy}{dx}\right) + (1 + x^2)\frac{d^2y}{dx^2} = \frac{2}{1 + x^2}$$

$$\Rightarrow (1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(x^2 + 1)\frac{dy}{dx} = 2$$ **Hence proved.**

14. Given: $x = a \sin^3 \theta$ and $y = b \cos^3 \theta$

Differentiating x and y w.r.t. θ,

We get $\frac{dx}{d\theta} = 3a \sin^2 \theta \cos \theta$

$$\frac{dy}{d\theta} = -3b \cos^2 \theta \sin \theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3b \cos^2 \theta \sin \theta}{3a \sin^2 \theta \cos \theta} = -\frac{b}{a} \cot \theta$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{b}{a} \text{cosec}^2 \theta \frac{d\theta}{dx}$$

$$= \frac{b}{a} \text{cosec}^2 \theta \cdot \frac{1}{3a \sin^2 \theta . \cos \theta}$$

$$= \frac{b}{3a^2} \sec \theta \ \text{cosec}^4 \theta$$

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right]_{\theta = \frac{\pi}{4}} = \frac{4\sqrt{2}.b}{3a^2}$$

15. Given, $y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

Put $x = \tan \theta$

$$\therefore \quad y = \cos^{-1}\left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}\right)$$

$$y = \cos^{-1} (\cos 2\theta)$$

$$y = 2\theta$$

$$\therefore \quad y = 2 \tan^{-1} x$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2 \cdot \frac{1}{1 + x^2}$$

or $$\frac{dy}{dx} = \frac{2}{1 + x^2}$$

16. Rewriting the expression as:

$$y = \cos^{-1} \frac{1 - 3^{2x}}{1 + 3^{2x}}$$

and puts $3^x = \tan t$.

Write the expression as:

$$y = \cos^{-1} \frac{1 - \tan^2 t}{1 + \tan^2 t} = \cos^{-1}(\cos 2t)$$

$\Rightarrow y = 2t \Rightarrow y = 2 \tan^{-1} 3^x$.

Now, $\frac{dy}{dx} = \frac{2}{1 + (3^x)^2} \times 3^x \ln(3)$.

$$\therefore \frac{dy}{dx} = \frac{2.3^x \ln 3}{1 + 3^{2x}}$$

Level - 2 ADVANCED COMPETENCY FOCUSED QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQs) (1 Mark)

1. Option (B) is correct.

Explanation: $v(t)$ is the velocity function. The derivative of velocity, $v'(t)$, gives the rate of change of velocity with respect to time. This rate of change of velocity is known as acceleration.

2. Option (B) is correct.

Explanation: $P'(x)$ gives the marginal profit. This is the rate at which profit changes per additional unit sold.

3. Option (B) is correct.

Explanation: sin(t) is continuous and differentiable for all real numbers t. The constant term 200 does not affect continuity or differentiability — it only shifts the graph vertically. Therefore, $T(t)$ is continuous and differentiable everywhere.

4. **Option (B) is correct.**

Explanation: For a function to be differentiable at a point, it must be continuous at that point. This is a necessary condition. It must have no sharp corners or cusps and no vertical tangents at that point. So, continuity is required, but being increasing, decreasing, or constant is not necessary for differentiability.

5. **Option (B) is correct.**

Explanation: The function $h(t) = 4t - t^2$ gives the height of water at time t. To find the rate at which the height is rising or falling, we need to find the derivative of $h(t)$, which gives the rate of change of height with respect to time. So,

$$h'(t) = \frac{d}{dt}(4t - t^2) = 4 - 2t$$

This expression tells us whether the water level is rising (positive value) or falling (negative value) at a given time t.

ASSERTION-REASON QUESTIONS (1 Mark)

1. **Option (A) is correct.**

Explanation: Assertion is true. If a velocity function is differentiable, it must also be continuous, as differentiability implies continuity.

Reason is also true because this is a fundamental concept in calculus. A function that is differentiable at a point is always continuous at that point.

Both assertion and reason are true and the reason is the correct explanation of assertion.

2. **Option (A) is correct.**

Explanation: Assertion is true because this function is a polynomial, and all polynomial functions are differentiable over the set of real numbers.

Reason is also true because this is a correct and well-known property of polynomials.

Both assertion and reason are true and the reason is the correct explanation of assertion.

3. **Option (A) is correct.**

Explanation: Assertion is true because at the point of a sharp corner (e.g., at x=0 for $|x|$), the function is continuous but not differentiable due to the sudden change in direction.

Reason is also true because differentiability requires the left-hand and right-hand derivatives to be equal. If they are not, the function is not differentiable at that point.

Both assertion and reason are true, and reason correctly explains assertion.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. With the help of the graphs, we observe the values of x for which $\cos^{-1}x$ attains integer values as:

$$\cos^{-1}x = 3$$
$$\Rightarrow \quad x = \cos 3$$
$$\cos^{-1}x = 2$$
$$\Rightarrow \quad x = \cos 2$$
$$\cos^{-1}x = 1$$
$$\Rightarrow \quad x = \cos 1$$

Thus, the points of discontinuity of the function $y = [\cos^{-1}x]$ are cos 3, cos 2 and cos 1.

2. The points in the domain [–3, 3] where the function is not differentiable are x = 0, 1 and (–1).

Because the function is not continuous at x = 0, it is not differentiable at x = 0.

Also, at x = 1 and (–1), the graph is pointed/not smooth, hence not differentiable.

3. Differentiating the given function as:

$$\frac{dy}{dx} = \frac{-1}{4x^4\sqrt{(4x^4)^2 - 1}}\frac{d(4x^4)}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-16x^3}{4x^4\sqrt{16x^8 - 1}} = \frac{-4}{x\sqrt{16x^8 - 1}}$$

4. Using the chain rule, we get

$$f'(x) = \frac{1}{2\sqrt{\cos^2 x - 25}} \times (-2\sin x \cos x)$$

On comparing, we get $g(x)$ = (–2 sin x cos x) or (– sin 2x).

5. Shyama is wrong.

Justifies by giving an example of a function whose first-order derivatives is the same as its second-order derivative.

For example, $f(x) = e^x$.

6. Given, $y = e^{\log \sin x}$

$$\Rightarrow \quad y = \sin x$$

Differentiates the above equation with respect to x as:

$$\frac{dy}{dx} = \cos x$$

(Award full marks if y (cot x) is obtained instead of cos x.)

7. False (F).

The function is not continuous in the domain of real numbers as $\frac{1}{x}$ is not defined at x = 0.

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. **(i)** The marginal cost is the derivative of the cost function with respect to x:

$$C'(x) = \frac{d}{dt}[5x^2 + 10x + 500] = 10x + 10$$

So, the marginal cost function is:
$C'(x)=10x+10$

(ii) Substitute $x=10$ into the marginal cost function:
$C'(10)=10(10)+10=100+10=$₹110

(iii) We use the difference in marginal costs:
Rate of change=$C'(11)-C'(10)$
$C'(11)=10(11)+10=110+10=$₹120
$C'(10)=$₹110′
Change=120 – 110=₹10

2. **(i)** Velocity is the first derivative of position:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(3t^3 - 6t^2 + 2t) = 9t^2 - 12t + 2$$

Velocity function:
$v(t)=9t^2-12t+2$

(ii) Acceleration is the derivative of velocity:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(9t^2 - 12t + 2) = 18t - 12$$

Now, put in $t=2$:
$a(2)=18(2) - 12 = 36 - 12 = 24\text{ m/s}^2$

(iii) Put $v(t)=0$:
$9t^2-12t+2=0$
Using the quadratic formula:

$$t = \frac{-(-12)\pm\sqrt{(-12)^2-4(9)(2)}}{2(9)} = \frac{12\pm\sqrt{144-72}}{18}$$

$$= \frac{12\pm\sqrt{72}}{18}$$

$$\Rightarrow t = \frac{12\pm 6\sqrt{2}}{18} = \frac{2\pm\sqrt{2}}{3}$$

Time when velocity is zero:

$$t = \frac{2+\sqrt{2}}{3}\text{ seconds and }\frac{2-\sqrt{2}}{3}\text{ seconds}$$

3. **(i)** Yes. The function is composed of sine and linear operations, both of which are continuous everywhere. Hence, the sum is also continuous.

(ii) Yes. Sine functions are differentiable everywhere. Scalar multiples and sums of differentiable functions remain differentiable.

(iii) We need the maximum value of $T'(t)$, the derivative of $T(t)$:

$$T(t) = 25 + 10\sin\left(\frac{\pi t}{12}\right)$$

Differentiating:

$$T'(t) = 10.\cos\left(\frac{\pi t}{12}\right).\frac{\pi}{12} = \frac{10\pi}{12}.\cos\left(\frac{\pi t}{12}\right)$$

$$= \frac{5\pi}{6}.\cos\left(\frac{\pi t}{12}\right)$$

Now since $-1 \le \cos\left(\frac{\pi t}{12}\right) \le 1$.

The maximum rate of change when $\cos\left(\frac{\pi t}{12}\right) = 1$

Maximum rate of change=$5\pi/6$ = C° per hour

4. **(i)** $E'(t) = \frac{d}{dt}(90 - 2t^2 + t) = -4t + 1$

(ii) $E''(t) = \frac{d}{dt}(-4t + 1) = -4$

(iii) Since $E'(t)=-4t+1$ is a linear function with negative slope, it decreases with t, so the maximum occurs at the lowest possible value of t in the domain.
If domain is not specified, we conclude:
Maximum of $E'(t)$ occurs when $t=0$
So, $E'(0)=-4(0)+1=1$
Therefore, rate of change of efficiency is maximum at $t=0$

CASE BASED QUESTIONS (4 Mark)

1. **(i)** $\frac{y}{x} = \tan 30° \Rightarrow y = \frac{x}{\sqrt{3}}$ or $x = \sqrt{3}y$

(ii) Putting y = 35 m in $y = 60 - 4.9t^2$, we have
$\Rightarrow 60 - 4.9t^2 = 35 \Rightarrow 4.9t^2 = 25$

$\Rightarrow t = \frac{5\sqrt{10}}{7}$ seconds.

(iii) (a) $x = \sqrt{3}y \Rightarrow x = 60\sqrt{3} - 4.9\sqrt{3}t^2$

$$\Rightarrow \left.\frac{dx}{dt}\right]_{t=\frac{5\sqrt{10}}{7}} = -4.9\sqrt{3}(2t)\Big]_{t=\frac{5\sqrt{10}}{7}} = -7\sqrt{30}\text{ m/s}$$

OR

(b) $\frac{dy}{dt} = -9.8t = -9.8t = -9.8\times 2 = -19.6$ m/s

Height of the sandbag is decreasing at the rate of 19.6 m/s

2. **(i)** Given $f(x) = a(x + 9)(x + 1)(x - 3)$
Put $f(x) = y = -1$ and $x = 0$, we get

$$-1 = a(-27) \Rightarrow a = \frac{1}{27}$$

(ii)

$$f(x) = \frac{1}{27}(x+9)(x+1)(x-3)$$

$$\Rightarrow f(x) = \frac{1}{27}(x^3+7x^2-21x-27)$$

$$f'(x) = \frac{1}{27}(3x^2+14x-21)$$

$$f''(x) = \frac{6x+14}{27}$$

$$f''(1) = \frac{20}{27}$$

3. (i) $$h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$$

Clearly $h(t)$ is a polynomial function, hence continuous.
Hence $h(t)$ is a continuous function.

(ii) For maximum height,

$$\frac{dh}{dt} = 0 \Rightarrow -7t + \frac{13}{2} = 0$$

$$\Rightarrow t = \frac{13}{14}$$

$$\frac{d^2h}{dt^2} = -7 < 0 \therefore \text{ height is maximum at } t = \frac{13}{14}$$

4. (i) R.H.D. of $f(x)$ at $(x = 1)$ $= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{|1+h-3|-|-2|}{h}$$

$$= \lim_{h\to 0} \frac{2-h-2}{h} = -1$$

(ii) L.H.D. of $f(x)$ at $(x = 1)$ $= \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0} \frac{\left[\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} - 2\right]}{-h}$$

$$= \lim_{h\to 0} \left[\frac{h^2 - 2h + 1 - 6 + 6h + 13 - 8}{-4h}\right]$$

$$= \lim_{h\to 0} \left[\frac{h^2 + 4h}{-4h}\right] = -1$$

(iii) (a) Since L.H.D. of $f(x)$ at $x = 1$ is same as R.H.D. of $f(x)$ at $x = 1$.
$f(x)$ is differentiable at $x = 1$.

OR

(b) $$f(x) = \begin{cases} x-3, & x \geq 3 \\ 3-x, & 1 \leq x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$$

$$[f'(x)]_{x=2} = 0 - 1 = -1$$

$$[f'(x)]_{x=-1} = \frac{2(-1)}{4} - \frac{3}{2} = -2$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. (i) Velocity is the first derivative of position:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(t^3 - 6t^2 + 9t + 100)$$

$a(t) = 3t^2 - 12t + 9$

Acceleration is the second derivative of position (or first derivative of velocity):

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(3t^2 - 12t + 9)$$

$a(t) = 6t - 12$

(ii) A particle comes to rest when velocity is 0, i.e., $v(t)=0$:
$3t^2-12t+9=0$
$t^2-4t+3=0$
$(t-1)(t-3)=0 \Rightarrow t=1$ or $t=3$
So, the particle comes to rest at $t=1$ s and $t=3$ s.

(iii) To check whether the motion is increasing or decreasing at $t=1$, we analyse the acceleration $a(t)=6t-12$ at $t=1$:
$a(1)=6(1)-12=-6$
Since acceleration is negative and velocity is zero at $t=1$ (from part ii), the velocity is decreasing, indicating that the particle is momentarily at rest and about to reverse direction.

2. (i) The function $\sin\left(\frac{\pi t}{12}\right)$ is continuous and dif-

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entiable for all real t because sine is a standard continuous and smooth (differentiable) function. A constant multiple or addition of such a function is also continuous and differentiable. Hence, $T(t)$ is continuous and differentiable on the closed interval [0, 24].

(ii) To find the rate of change, we differentiate $T(t)$:

$$T'(t) = \frac{d}{dt}\left[25 + 10\sin\left(\frac{\pi t}{12}\right)\right] = 10.\cos\left(\frac{\pi t}{12}\right).\frac{\pi}{12}$$

$$T'(t) = \frac{10\pi}{12}\cos\left(\frac{\pi t}{12}\right) = \frac{5\pi}{6}\cos\left(\frac{\pi t}{12}\right)$$

Substituting at $t = 6$:

$$T'(6) = \frac{5\pi}{6}.\cos\left(\frac{\pi.6}{12}\right) = \frac{5\pi}{6}.\cos\left(\frac{\pi}{2}\right)$$

$$= \frac{5\pi}{6} \times 0 = 0$$

At $t = 18$

$$T'(18) = \frac{5\pi}{6}.\cos\left(\frac{18\pi}{12}\right) = \frac{5\pi}{6}.\cos\left(\frac{3\pi}{2}\right) = \frac{5\pi}{6}$$

$\times 0 = 0$

So, the rate of change of temperature at both $t=6$ and $t=18$ is 0°C/hour, i.e., temperature is at a turning point.

(iii) To find maximum rate of increase, we maximise

$$T'(t) = \frac{5\pi}{6\cos}\left(\frac{\pi t}{12}\right)$$

Max of $\cos(\theta)$ is 1, which occurs when:

$$\frac{\pi t}{12} = 0 \Rightarrow t = 0$$

So, temperature is increasing the fastest at $t=0$ with:

$$T'(0) = \frac{5\pi}{6}.\cos(0) = \frac{5\pi}{6}$$