CHAPTER 6

Application of Derivatives

Level - 1 CORE SUBJECTIVE QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ) (1 Marks)

1. **Option (C) is correct.**

   Explanation: $y = e^x \Rightarrow \frac{dy}{dx} = e^x$

   In the domain ($\mathbb{R}$) of the function, $\frac{dy}{dx} > 0$, hence the function is strictly increasing in $(-\infty, \infty)$.

2. **Option (B) is correct.**

3. **Option (B) is correct.**

   Explanation:

   Given the function:

   $$f(x) = x^3 - 3x^2 + 12x - 18$$

   Differentiate:

   $$f'(x) = 3x^2 - 6x + 12$$

   Solve $f'(x) = 0$:

   $$3x^2 - 6x + 12 = 0$$
   $$\Rightarrow \quad 3(x^2 - 2x + 4) > 0$$
   $$\Rightarrow \quad x^2 - 2x + 4 > 0$$
   $$\Rightarrow \quad x^2 - 2x + 1 + 3) > 0$$
   $$\Rightarrow \quad (x-1)^2 + 3 > 0$$

   Since $f'(x)$ is always non-negative for all x, $f(x)$ is strictly increasing on $\mathbb{R}$.

4. **Option (B) is correct.**

   Explanation: Given: The side length of a square is decreasing at a rate of $\frac{ds}{dt} = -1.5$ cm/s.

   Perimeter of a square:

   $$P = 4s$$

   Differentiate both sides w.r.t. time t:

   $$\frac{dP}{dt} = 4\frac{ds}{dt}$$
   $$\frac{dP}{dt} = 4(-1.5) = -6 \text{ cm/s}$$

   The negative sign indicates a decrease.

5. **Option (C) is correct.**

   Explanation: For Local Maximum

   $f'(x) = 0$ at the point.

   $f'(x)$ changes sign from $+$ to $-$ (increasing to decreasing).

   This means the function rises before the point and falls after it.

   For Local Minimum

   $f'(x) = 0$ at the point.

   $f'(x)$ changes sign from $-$ to $+$ (decreasing to increasing).

   This means the function falls before the point and rises after it.

   For Inflection Point

   $f''(x)$ changes sign

   $f'(x)$ may be 0 but does not change sign:

   If $f'(x) > 0$, it stays $+$ (increasing function).

   If $f'(x) < 0$, it stays $-$ (decreasing function).

   Thus, at an inflection point, $f'(x)$ does not change sign (stays + to + or – to –), but $f''(x)$ changes sign.

6. **Option (C) is correct**

   Explanation: Given the surface area of a sphere:

   $$S = 4\pi r^2$$

   Differentiate with respect to r :

   $$\frac{dS}{dr} = 8\pi r$$

   Put $r = 4$:

   $$\frac{dS}{dr} = 8\pi(4) = 32\pi \text{ cm}^2/\text{cm}$$

7. **Option (A) is correct**

   Explanation: Given $f(x) = kx - \sin x$ differentiate:

   $$f'(x) = k - \cos x$$

   For $f(x)$ to be strictly increasing:

   $$f'(x) > 0 \Rightarrow k - \cos x > 0 \Rightarrow k > \cos x$$

   Since $\cos x \leq 1$, we need $k > 1$.

8. **Option (A) is correct**

   Explanation: $f(x) = \frac{x}{2} + \frac{2}{x}$

   First derivative:

   $$f'(x) = \frac{1}{2} - \frac{2}{x^2}$$

   Put $f'(x) = 0$:

   $$\frac{1}{2} = \frac{2}{x^2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Second derivative:

$$f''(x) = \frac{4}{x^3}$$

Since, $f''(2) = \frac{1}{2} > 0$ (local minimum)

and $f''(-2) = -\frac{1}{2} < 0$ (local maximum)

Thus, the function has a local minimum at $x = 2$.

9. Option (A) is correct

Explanation: The slope of the curve is given by the derivative $\frac{dy}{dx}$.

Given $y = 7x - x^3$, then

$$\frac{dy}{dx} = 7 - 3x^2$$

slope is changing with respect to time,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(7 - 3x^2\right) = -6x\frac{dx}{dt}$$

We're given that $\frac{dx}{dt} = 2$ units per second and $x = 5$.

Substitute these values into the expression:

$$-6x\frac{dx}{dt} = -6(5)(2) = -60$$

Therefore, the rate at which the slope of the curve is changing is -60 units/sec.

10. Option (D) is correct

Explanation: $f'(x) = 3x^2$

$f'(0) = 0$

$f''(x) = 6^2$

$f''(0) = 0$

So, $x = 0$ will be point of inflection for the given function $f(x) = x^3$.

11. Option (B) is correct

Explanation: Given

$$\frac{dr}{dt} = 0.5 \text{ cm/s}$$

$$C = 2\pi r$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

$$= 2\pi \times 0.5 = \pi \text{ cm/s}$$

12. Option (C) is correct

Explanation:

Given: $f(x) = x^3 - 3x^2 + 3x$

Derivative: $f'(x) = 3x^2 - 6x + 3$

$= 3(x-1)^2 \geq 0$

Since $f'(x)$ is always non – negative, $f(x)$ is increasing for all $x \in \mathbb{R}$.

13. Option (C) is correct

Explanation: (Strictly Decreasing Condition)

Given: $f(x) = a(x - \cos x)$

Derivative: $f'(x) = a(1 + \sin x)$

For $f(x)$ to be strictly decreasing, $a(1 + \sin x) < 0$ for all x.

Since $1 + \sin x \geq 0$, a must be negative.

$a \in (-\infty, 0)$.

14. Option (B) is correct

Explanation: Given function:

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$

$$f'(x) = \frac{d}{dx}\left(2x^3 + 9x^2 + 12x - 1\right)$$

$$= 6x^2 + 18x + 12$$

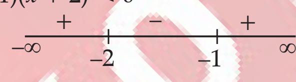
For decreasing intervals,

$$f'(x) < 0$$

$$6x^2 + 18x + 12 < 0$$

$$x^2 + 3x + 2 < 0$$

$$(x+1)(x+2) < 0$$



Solving $(x + 1)(x + 2) < 0$,

So, $f(x)$ is decreasing in $(-2, -1)$.

15. Option (C) is correct

Explanation: Given function:

$$f(x) = x^3 + 3x$$

$$f'(x) = \frac{d}{dx}(x^3 + 3x)$$

$$= 3x^2 + 3$$

Since:

$$3x^2 + 3 > 0 \text{ for all } x \in \mathbb{R}$$

(Quadratic expression with positive coefficient and no real roots) $f'(x) > 0$ always, meaning $f(x)$ is increasing for all real numbers.

16. Option (A) is correct

Explanation: Given:

$$\theta = 2\sin\theta$$

Using differentiation:

$$\frac{d}{dt}(\sin\theta) = \cos\theta\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = 2\cos\theta\frac{d\theta}{dt}$$

$$1 = 2\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

ASSERTION-REASON QUESTIONS (1 Marks)

1. Option (A) is correct

Explanation: Given: Side of a square increases at 0.2 cm/s.

Perimeter of a square:

$$P = 4s$$

Differentiating w.r.t. time t :

$$\frac{dP}{dt} = 4\frac{ds}{dt} = 4(0.2) = 0.8 \text{ cm / s}$$

So, both A and R are true, and R is the correct explanation of A.

2. Option (C) is correct

Explanation: Given function:

$$f(x) = (\cos^{-1}x)^2$$

Range of $\cos^{-1}x$ is $[0, \pi]$

Maximum value occurs at $x = -1$:

$$(\cos^{-1}(-1))^2 = \pi^2$$

So, Assertion (A) is True, but Reason (R) is False.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. The marginal cost function is

$$C'(x) = 0.00039x^2 + 0.004x + 5.$$

$$C'(150) = ₹\ 14.375$$

2. $$f'(x) = 12x^2 - 36x + 27$$

$$= 3(2x-3)^2 \geq \text{ for all } x \in \mathbb{R}$$

$\therefore f$ is increasing on $\mathbb{R}$.

Hence $f(x)$ does not have maxima or minima.

3. $$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}}$$

For critical points, $f'(x) = 0$

$$\Rightarrow 16x = 2 \Rightarrow x = \frac{1}{8}$$

x	$f(x)$
0	0
$\frac{1}{8}$	$\frac{-9}{4}$ (Absolute minimum)
1	6 (Absolute maximum)

4. $$f(x) = 4x^2 + \frac{1}{x} (x \neq 0)$$

$$f'(x) = 8x - \frac{1}{x^2} = 0$$

$$\Rightarrow \quad x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2}$$

$$f''(x) = 8 + \frac{2}{x^3} > 0 \text{ at } x = \frac{1}{2}$$

$$\therefore \text{ Local minimum value } = f\left(\frac{1}{2}\right) = 3$$

5. $$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x+1)(x-1)}{x^2}$$

$$f'(x) = 0 \Rightarrow x = -1, 1$$

$$f''(x) = \frac{2}{x^3} \Rightarrow f''(-1) = -2 < 0 \text{ and } f''(1) = 2 > 0$$

$\therefore -1$ is a point of local maximum

The local maximum value $= f(-1) = -2 = M$

$$f''(1) = 2 > 0$$

$\therefore$ 1 is point of local minimm

The local minimum value $= f(1) = 2 = m$

So, $\quad M - m = -4$

6. $$f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2}$$

$$= e^x + \frac{1}{e^x} + \frac{x^2}{1+x^2} > 0 \text{ for all } x \in \mathbb{R}$$

$\therefore f$ is strictly increasing over its domain $\mathbb{R}$.

7. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$\Rightarrow 4x^2(x-3) < 0$ for $x < 3, x \neq 0$

$\Rightarrow f'(x) < 0$ for $x < 3, x \neq 0$

Thus, $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing on $(-\infty, 0) \cup (0, 3)$.

8. Given, $\frac{dV}{dt} = 6\text{ cm}^3/\text{sec}$. Since, $V = x^3$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 6 = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2}{x^2} \text{ cm/sec}$$

[Here, x be the edge of cube]

Now, Surface Area $= S = 6x^2$

$$\Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} = 12 \times 8 \times \frac{2}{(8)^2} = 3\text{ cm}^2/\text{sec}$$

9. $$f(x) = \frac{4\sin x}{2+\cos x} - x$$

$$\Rightarrow \quad f'(x) = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

when $x \in \left[0, \frac{\pi}{2}\right], \cos x > 0 \Rightarrow \frac{\cos x(4-\cos x)}{(2+\cos x)^2} \geq 0$

Hence, $f(x)$ is increasing.

10. Let $C = 2\pi r$, be the circumference of the circle,

Area of circle, $A = \pi r^2$

Given, $\frac{d(A)}{dt} = 2$

$$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{\pi r} \text{ cm/sec}$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = \frac{2}{r} = \frac{2}{5} \text{ cm/sec at } r = 5\text{ cm}$$

11. Let 'V' be the volume, 'h' and 'r' be the height and radius of the cone,

Given, $\quad h = \frac{r}{3}, \frac{dV}{dt} = 15\text{ cm}^3/\text{min}$

$$\because \quad V = \frac{1}{3}\pi r^2 h = 3\pi h^3$$

$$\Rightarrow \quad \frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \quad 15 = 9\pi \times (4)^2 \frac{dh}{dt} \qquad [\because h = 4]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{15}{9\pi \times 16} = \frac{5}{48\pi} \text{ cm/min}$$

12. Let edge of cube be x cm.

$$S = 6x^2, \frac{dS}{dt} = 72 \text{ cm}^2/\text{sec}$$

$$\therefore \quad \frac{dS}{dt} = 12x$$

Given,

$$\Rightarrow \quad 12x\frac{dx}{dt} = 72 \Rightarrow \frac{dx}{dt} = \frac{6}{x}$$

Volume, $V = x^3$

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$= 3 \times x^2 \times \frac{6}{x} = 18x$$

$$\left.\frac{dV}{dt}\right]_{x=3} = 54 \text{ cm}^3/\text{sec}$$

$\therefore$ Volume is increasing at the rate of 54 cm^3/sec.

13. $$f'(x) = \frac{1 - \log x}{x^2},$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad \log x = 1 \Rightarrow x = e$$

$$f''(x) = \frac{2x\log x - 3x}{x^4}$$

$$\Rightarrow \quad f''(e) = -\frac{1}{e^3} < 0$$

i.e. $x = e$ is a point of local maximum.

14. $$\frac{dr}{dt} = 0.5 \text{ cm/s (given)}$$

Now, $$S = 4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r.\frac{dr}{dt}$$

$$\therefore \quad \left.\frac{dS}{dt}\right]_{x=1.5 \text{ cm}} = 8\pi(1.5)(0.5)$$

$$= 6\pi \text{ cm}^2/\text{s}$$

15. $$f'(x) = \frac{16[4+\cos x]\cos x + 16\sin^2 x}{(4+\cos x)^2} - 1$$

$$= \frac{\cos x(56 - \cos x)}{(4+\cos x)^2}$$

In $\left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0 \Rightarrow f'(x) < 0$

$\therefore f(x)$ in strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

16. Let numbers be x and $\frac{9}{x}$.

$$\therefore \quad \text{Required sum} = x^2 + \frac{81}{x^2} = f(x) \text{ say}$$

$$f'(x) = 2x - \frac{162}{x^3}$$

$$f'(x) = 0 \Rightarrow x = 3$$

$$f''(x) = 2 + \frac{486}{x^4}$$

At $x = 3$

$$f''(x) > 0$$

$\therefore f(x)$ is minimum at $x = 3$.

Thus, numbers are 3 and 3.

$\therefore$ Sum is minimum when both numbers are 3.

17. $$f'(x) = \frac{1}{2+x} - \frac{2}{(2+x)^2} = \frac{x}{(2+x)^2}$$

Sign of $f'(x)$

$\therefore f(x)$ is decreasing in $(-2, 0)$ and increasing in $(0, \infty)$

18. Differentiating equation $3y = ax^3 + 1$ with respect to 'x',

$$3\frac{dy}{dx} = 3ax^2$$

Putting $x = 1, \frac{dy}{dx} = 2$, we get $3(2) = 3a(1)^2 \Rightarrow a = 2$

19. Let 'r' be the radius, C the circumference and A the area of the circle.

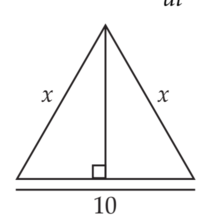
Then, $\frac{dC}{dt} = k$ (Constant), also $C = 2\pi r$

$$\Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{k}{2\pi}$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r \cdot \frac{k}{2\pi} = kr,$$

$\therefore$ The rate of change of area is directly proportional to its radius.

20. Let the equal side be 'x', then $\frac{dx}{dt} = 4$ cm/s



$$A = 5\sqrt{x^2 - 25} \Rightarrow \frac{dA}{dt} = \frac{5x}{\sqrt{x^2-25}} \cdot \frac{dx}{dt}$$

$$\left.\frac{dA}{dt}\right]_{x=10} = \frac{40}{\sqrt{3}} \text{ cm}^2/\text{s}$$

21. $$f'(x) = a(\sec^2 x + \text{cosec}^2 x)$$

As $a > 0$ and $\sec^2 x$, $\text{cosec}^2 x$ are squares, $f'(x) > 0$

$\therefore$ $f(x)$ is an increasing function in its domain.

22. The given statement is "True".

$$f'(x) = -\frac{b}{x^2}$$

For $b < 0$, $f'(x) > 0$ in $(-\infty, 0)$ and $(0, \infty)$

$\therefore$ $f(x)$ is strictly increasing in both these intevals.

23. Here, $$f(x) = 2x^3 - 3x$$

$$f'(x) = 6x^2 - 3 = 3(2x^2 - 1)$$

Now, $f'(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$f(x)$ is strictly increasing in $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

24. We know that, $-1 \le \sin 2x \le 1$

$\Rightarrow -1 + 5 \le \sin 2x + 5 \le 1 + 5$

$\Rightarrow 4 \le \sin 2x + 5 \le 6$

So, maximum value is 6 and minimum value is 4.

25. Here, $\frac{dx}{dt} = \frac{dy}{dt}$

Given $y^2 = 8x$ gives $2y\frac{dy}{dt} = 8\frac{dx}{dt}$

$\Rightarrow 2y = 8$ or $y = 4$

Also, $y = 4$ gives $x = 2$

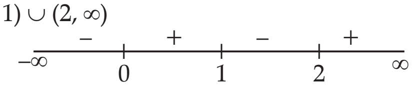
Thus, the required point is (2, 4).

26. $$f'(x) = 4x^3 - 12x^2 + 8x$$
$$= 4x(x-1)(x-2)$$

$f'(x) = 0$ gives $x = 0, 1, 2$

For strictly increasing, $f'(x) > 0$

$x \in (0, 1) \cup (2, \infty)$



27. $$f'(x) = 3x^2 - 24x + 36$$
$$= 3(x-2)(x-6)$$

f is strictly increasing, $f'(x) > 0$

$3(x-2)(x-6) > 0 \Rightarrow x \in (-\infty, 2) \cup (6, \infty)$

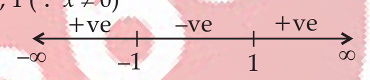
28. $$f'(x) = 3x^2 - \frac{3}{x^4}$$

$$f'(x) = 0$$

$$\Rightarrow \frac{3(x^6 - 1)}{x^4} = 0$$

$$\Rightarrow \frac{3(x^3-1)(x^3+1)}{x^4} = 0$$

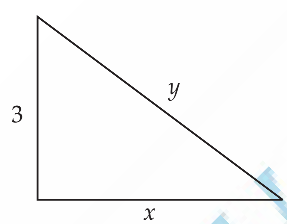
$\Rightarrow x = -1, 1$ ($\because x \neq 0$)



$\therefore$ $f'(x)$ is decreasing when $x \in (-1, 1) - \{0\}$

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1.



Here, $$x^2 + 3^2 = y^2 \quad \text{...(i)}$$

When $y = 5$ then $x = 4$,

On differentiating eq. (ii) w.r.t. t, $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$$\Rightarrow \quad 4(200) = 5\frac{dy}{dt} \quad \left[\because \frac{dx}{dt} = 200 \text{ cm/s}\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = 160 \text{ cm/s}$$

2. $A = \frac{1}{3}\sqrt{t} \therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}; \forall\, t \in [5, 18]$

Differentiating again w.r.t. t, we get $\frac{d^2A}{dt^2} = -\frac{1}{12t^{3/2}} < 0$,

$t \in [5, 18]$

This means that the rate of change of the ability to understand spatial concepts decreases (slows down) with age.

3. Given, $$f(x) = \frac{\log x}{x}$$

$$\Rightarrow \quad f'(x) = \frac{1 - \log x}{x^2}; x > 0$$

For strictly increasing,

$$f'(x) > 0$$

$$\Rightarrow \quad \frac{1 - \log x}{x^2} > 0$$

$$\Rightarrow \quad \log x < 1$$

$$\Rightarrow \quad x < e \text{ and } x > 0$$

Thus, $f(x)$ is strictly increasing, $x \in (0, e)$

For strictly decreasing,

$$f'(x) < 0$$

$$\Rightarrow \quad \frac{1 - \log x}{x^2} < 0$$

$$\Rightarrow \quad 1 - \log x < 0$$

$$\Rightarrow \quad \log x > 1$$

$$\Rightarrow \quad x > e$$

$\therefore$ $f(x)$ is strictly decreasing in (e, ∞).

4. $$f(x) = \frac{x}{2} + \frac{2}{x}; x \in [1, 2]$$

$$\Rightarrow \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$$

For absolute maximum/minimum, put $f'(x) = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = 2$

Now, $f(1) = \frac{5}{2}$ and $f(2) = 2$

$\therefore$ Absolute maximum value $= \frac{5}{2}$ and absolute minimum value $= 2$

5. $$f(x) = \sin 3x$$

$$\Rightarrow \quad f'(x) = 0$$

$$\Rightarrow \quad 3\cos 3x = 0$$

$$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}$$

$f'(x) \geq 0, \forall x \in \left[0, \frac{\pi}{6}\right] \Rightarrow f(x)$ is increasing on $\left[0, \frac{\pi}{6}\right]$

$f'(x) \leq 0, \forall x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow f(x)$ is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. $$f(x) = x^4 - 62x^2 + ax + 9$$

$$\Rightarrow \quad f'(x) = 4x^3 - 124x + a$$

As at $x = 1$, f attains local maximum value, $f'(1) = 0$

$\Rightarrow a = 120$

Now,
$$f'(x) = 4x^3 - 124x + 120$$
$$= 4(x-1)(x^2 + x - 30)$$
$$= 4(x-1)(x-5)(x+6)$$

Critical points are $x = -6, 1, 5$

$$f''(x) = 12x^2 - 124$$

$f''(-6) > 0, f''(1) < 0, f''(5) > 0$

so f attains local maximum value at $x = 1$ and local minimum value at $x = -6, 5$

2. Let length of rectangle be x cm and breadth be $(150 - x)$ cm.

Let r be the radius of cylinder. So, circumference of base,

$$2\pi r = x \Rightarrow r = \frac{x}{2\pi}$$

$$V = \pi r^2 h = \pi\left(\frac{x^2}{4\pi^2}\right)(150 - x)$$

$$= \frac{75x^2}{2\pi} - \frac{x^3}{4\pi}$$

$$\frac{dV}{dx} = \frac{150x}{2\pi} - \frac{3x^2}{4\pi}$$

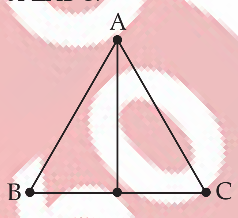
$$\frac{d^2V}{dx^2} = \frac{150}{2\pi} - \frac{6x}{4\pi}$$

$$\frac{dV}{dx} = 0 \Rightarrow x = 100 \text{ cm}$$

$\left.\frac{d^2V}{dx^2}\right|_{x = 100\text{ cm}} = -\frac{75}{\pi} < 0 \Rightarrow V$ is maximum when $x = 100$ cm.

Length of rectangle is 100 cm and breadth of rectangle is 50 cm.

3. In an equilateral triangle, median is same as altitude. Let 'h' denote the length of the median (or altitude) and 'x' be the side of ΔABC.



Then, $h = \frac{\sqrt{3}}{2}x$ or $x = \frac{2h}{\sqrt{3}}$ ...(i)

It is given that $\frac{dh}{dt} = 2\sqrt{3}$.

So, by (i) we have

$$\frac{dx}{dt} = \frac{2}{\sqrt{3}}\frac{dh}{dt} \Rightarrow \frac{dx}{dt} = 4$$

Thus, the side of ΔABC is increasing at the rate of 4 cm/sec.

4. Let the two numbers be x and y. Then, $x + y = 5$ or $y = 5 - x$

Let S denote the sum of the cubes of these numbers. Then

$$S = x^3 + y^3$$
$$= x^3 + (5-x)^3$$
$$\frac{dS}{dx} = 3x^2 - 3(5-x)^2$$
$$= 15(2x - 5)$$

Now $\frac{dS}{dx} = 0$, gives $x = \frac{5}{2}$

$$\frac{d^2S}{dx^2} = 30 > 0$$

$\therefore S$ is minimum at $x = \frac{5}{2}$

So, the two numbers are $\frac{5}{2}$ and $\frac{5}{2}$

Thus, $x^2 + y^2 = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}$

Level - 2 ADVANCED COMPETENCY FOCUSED QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQs) (1 Marks)

1. **Option (C) is correct.**

Explanation: Let the length be l and breadth be b.

Given, Total fencing (Perimeter) = 100 meters

$\Rightarrow \quad 2l + 2b = 100$

$\Rightarrow \quad l + b = 50$

$\Rightarrow \quad b = 50 - l$

Now, Area $A = l \times b = l(50 - l) = 50l - l^2$

To maximise area, differentiate A with respect to l:

$$\frac{dA}{dl} = 50 - 2l$$

Set derivative to zero:

$$50 - 2l = 0 \Rightarrow l = 25$$

So, $b = 50 - 25 = 25$

Shape is a square with sides of 25 m.

2. **Option (B) is correct.**

Explanation: Volume of cone:

$$V = \frac{1}{3}\pi r^2 h$$

But here, as water rises, r and h are variables, but since the tank is a cone, they are related via similar triangles:

$$\frac{r}{h} = \frac{3}{12} = \frac{1}{4} \Rightarrow r = \frac{h}{4}$$

Substitute into volume formula:

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

$$= \frac{1}{3}\pi.\frac{h^2}{16}.h = \frac{\pi h^3}{48}$$

Differentiating both sides:

$$\frac{dV}{dt} = \frac{\pi}{48}.3h^2.\frac{dh}{dt} = \frac{\pi h^2}{16}.\frac{dh}{dt}$$

Now substitute:

$$\frac{dV}{dt} = 2 \text{ and } h = 4$$

$\therefore \quad 2 = \frac{\pi(4)^2}{16}.\frac{dh}{dt} = \frac{16\pi}{16}.\frac{dh}{dt}$

$\Rightarrow \quad 2 = \pi . \frac{dh}{dt}$

$\Rightarrow \quad \frac{bh}{dt} = \frac{2}{\pi}$ m/min

3. **Option (B) is correct.**

Explanation: Velocity is the derivative of position:

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

Speed is the magnitude of velocity:

$$\text{Speed} = |v(t)|$$

At $t = 0$:

$v(0) = 3(0)^2 - 12(0) + 9 = 9$

$\Rightarrow \quad |v(0)| = 9$

At $t = 1$:

$v(1) = 3(1)^2 - 12(1) + 9 = 0$

$\Rightarrow \quad |v(1)| = 0$

At $t = 2$:

$v(2) = 3(4) - 12(2) + 9$

$= 12 - 24 + 9 = -3$

$\Rightarrow \quad |v(2)| = 3$

At $t = 3$:

$v(3) = 3(9) - 12(3) + 9$

$= 27 - 36 + 9 = 0$

$\Rightarrow \quad |v(3)| = 0$

Maximum speed = 9 m/s at $t = 0$

4. **Option (A) is correct.**

Explanation: Given, profit function, $P(x) = -2x^2 + 80 - 500$

$\therefore \quad P'(x) = -4x + 80$

and $\quad P''(x) = -4$

For maximum profit , put $P'(x) = 0$

i.e., $-4x + 80 = 0 \Rightarrow x = 20$

At $x = 20$, $P''(x) < 0$, Thus maximum.

5. **Option (A) is correct.**

Explanation: Given, Length of ladder = 10 m (constant, hypotenuse)

At a certain moment, the bottom of the ladder is 6 m from the wall

Bottom is sliding away from the wall at $\frac{dx}{dt} = 1$ m/s

We need to find the rate at which the top is sliding down $\frac{dy}{dt}$ when $x = 6$

Let: $x(t)$ be distance of bottom from the wall and $y(t)$ be height of the top from the ground.

By Pythagoras:

$$x^2 + y^2 = 10^2 = 100$$

Differentiating both sides w.r.t. time t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$x = 6 \text{ and } \frac{dx}{dt} = 1$$

Find y using Pythagoras:

$$y = \sqrt{100 - 36} = \sqrt{64} = 8$$

Now substitute:

$$6(1) + 8\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{6}{8} = -\frac{3}{4}$$

Thus, top is descending at rate $\frac{3}{4}$ m/s.

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (A) is correct.**

   Explanation: Assertion is true. The derivative of velocity is acceleration. A car reaches maximum speed when velocity is maximum, which occurs when the derivative of velocity (i.e., acceleration) is zero and the sign of acceleration changes from positive to negative.

   Reason is also true.This is a standard result in calculus — local maxima or minima of a differentiable function occur where its first derivative is zero.

2. **Option (A) is correct.**

   Explanation: Assertion is true. Companies often model profit as a function of quantity. To maximise profit, they find critical points (where the first derivative is zero) and then use the second derivative test to confirm if it's a maximum.

   Reason is also true and correctly explains the role of the second derivative:

3. **Option (A) is correct.**

   Explanation: Assertion is true. If the water level is going down, then the rate of change of height with respect to time (i.e., the derivative) is negative.

   Reason is also true. A function is decreasing in an interval where its derivative is less than zero.

4. **Option (A) is correct.**

   Explanation: Assertion is true. As the ladder slides, the angle between the ladder and the ground (say θ) decreases because the base of the triangle increases and the height decreases.

   Reason is also true. The triangle becomes flatter as the base (distance from wall) increases and height decreases, which reduces the angle between the ladder and the ground.

   Both assertion and reason are true and reason is the correct explanation of assertion.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. Differentiate s with respect to time to find the velocity of the drone as:

   $$s' = nt^{(n-1)}$$

   Differentiate s' with respect to time to find the acceleration of the drone as:

   $$s'' = n(n-1)t^{(n-2)}$$

   Equate velocity of the drone to its acceleration at 3 seconds to find n as:

   $$n \times 3^{(n-1)} = n(n-1) \times 3^{(n-2)}$$

   $$\Rightarrow \quad n \times 3^{(n-1)} = n(n-1) \times 3^{(n-1)} \times 3^{-1}$$

   $$\Rightarrow \quad n = 4$$

2. The area of a circle in terms of diameter, D as:

   $$A = \frac{\pi}{4}D^2$$

   The rate of change of area with respect to diameter as follows:

   $$\frac{dA}{dD} = \frac{\pi}{2}D$$

   When $D = 8$ cm, $\frac{dA}{dD} = \frac{\pi}{2} \times 8 = 4\pi$ cm^2/cm.

   Concludes that for a small change in diameter, the area changes by a factor of 4π.

   (Award full marks if the problem is solved correctly using approximation concept to obtain $4\pi\, x$ as the answer, where x is the small change in diameter.)

3. Let the rate of change of area of the circular plate with respect to time is equal to a constant k.

   Since, $A = \pi r^2$

   $\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt} = k$ cm^2/s, where k is a positive real number, t is the time, A and r are area and radius of the circular plate respectively.

   The rate of change of perimeter of the circular plate with respect to time iss:

   $\frac{dP}{dt} = 2\pi \times \frac{dr}{dt}$, where $P = 2\pi r$ is the perimeter of the circular plate.

   $$\therefore \quad \frac{dP}{dt} = 2\pi \times \frac{k}{2\pi r} = \frac{k}{r}$$

   Concludes that Milind's claim is correct.

4. The volume of a sphere, V, is given by $\frac{4}{3}\pi r^3$, where r is the radius of the sphere. Differentiating with respect to time, we get

   $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad ...(i)$$

   The surface area of a sphere A, is given by $4\pi r^2$

   Differentiating the same with respect to time, we get

   $$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \quad ...(ii)$$

   From eqs. (i) & (ii) the relation between $\frac{dV}{dt}$ and $\frac{dA}{dt}$ is given as:

   $$\frac{dV}{dt} = \frac{r}{2} \times \frac{dA}{dt} \quad ...(iii)$$

   Substituting $r = 1$ in the eq. (iii), we get $2\frac{dV}{dt} = \frac{dA}{dt}$.

5. Differentiate s to find velocity as:

   $$s' = A \cos t - B \sin t$$

   Differentiate s' to find acceleration as:

   $$s'' = -A \sin t - B \cos t$$

   Accleration in terms of distance as $s'' = (-s)$ and the magnitudes of distance and acceleration are the same. Hence, concludes that acceleration is always numerically equal to the distance of the particle from the fixed point.

6. The expression for the rate of change of the area of the dosa as:

$$\frac{dA}{dt} = \pi \times 2r \times \frac{dr}{dt}\ \text{cm}^2/\text{s} \qquad [\because A = \pi r^2]$$

Using the given information we find the rate of change of the area of the dosa when its radius is 9 cm is $\frac{dA}{dt}$

$= \pi \times 2 \times 9 \times 2 = 36\pi\ \text{cm}^2/\text{s}$.

7. The rate of change of the affected area as $\frac{dA}{dt}$

$$= 2\pi r \frac{dr}{dt}\ \text{km}^2/\text{sec}. \qquad [\because A = \pi r^2]$$

The rate of change of the area affected by the earthquake when the radius of the affected area (r) is 25 km as $2 \times 3.14 \times 25 \times 6 = 942\ \text{km}^2/\text{sec}$.

8. $f(x)$ need not be an increasing function on [0, 5].

Gives a reason. For example, even through $f'(5) > 0$, there may be an x in (4, 5), such that $f'(x) < 0$.

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. The volume of the air in the basketball $= \frac{4}{3}\pi r^3$ and the rate of loss of volume of air $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

The surface area of the basketball is $4\pi r^2$ and the rate of loss of the surface area $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$.

From above, the ratio $= r : 2$ and at $r = 8$ cm, the ratio is 8 : 2 or 4 : 1.

2. The volume function of the cylindrical disk $= V = \pi \times R^2 \times H$.

Since volume remains constant, the derivative of V should be zero.

$$\frac{dV}{dt} = 0$$

Applies chain rule to get the equation as:

$$\pi \times R^2 \times \frac{dH}{dt} + \pi \times H \times 2R \times \frac{dR}{dt} = 0$$

Simplifies the equation in step 3 to get the ratio as:

$$\frac{dH}{dt} : \frac{dR}{dt} = \frac{-2H}{R}$$

Substitutes $H = \frac{V}{\pi R^2}$ to get the ratio in terms of R as:

$$\frac{dH}{dt} : \frac{dR}{dt} = \frac{-2V}{\pi R^3}$$

3. Assumes the perimeter of one square as x m and the perimeter of the other square as $(a - x)$ m.

The combined area of the two squares as:

$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{a-x}{4}\right)^2 \text{m}^2$$

Differentiates the combined area as:

$$\frac{dA}{dt} = \frac{(4x - 2a)}{16}$$

Equates $\frac{dA}{dt}$ to 0 to find the critical point as $x = \frac{a}{2}$ m.

The second derivatives is:

$$\frac{d^2A}{dx^2} = \frac{1}{4} > 0$$

Concludes that $x = \frac{a}{2}$ m is a minima.

Hence, the combined area of the two squares will be minimum when the side lengths of both the squares is $\frac{a}{8}$ m.

4. The first derivative of the function is given as:

$$\frac{dp}{dt} = 16 - 2t.$$

Obtain the critical point as $t = 8$ by equating $16 - 2t$ to 0.

The second derivative is given as:

$$\frac{d^2p}{dt^2} = -2$$

This means that the function $p(t)$ will be at its maximum at $t = 8$.

The maximum price of the share as $p(8) = 16 \times 8 - 8^2 + 8 =$ ₹ 72.

CASE BASED QUESTIONS (4 Marks)

1. **(i)** $V = (40 - 2x)(25 - 2x)\,x\ \text{cm}^3$

(ii) $\frac{dV}{dx} = 4(3x - 50)(x - 5)$

(iii) (a) For extreme values $\frac{dV}{dx} = 4(3x - 50)(x - 5) = 0$

$$\Rightarrow x = \frac{50}{3} \text{ or } x = 5$$

$$\frac{d^2V}{dx^2} = 24x - 260$$

$\therefore \frac{d^2V}{dx^2}$ at $x = 5$ is $-140 < 0$

$\therefore V$ is max when $x = 5$

OR

(b) For extreme values $\frac{dV}{dx} = 4(3x^2 - 65x + 250)$

$$\frac{d^2V}{dx^2} = 4(6x - 65)$$

$\frac{dV}{dx}$ at $x = \frac{65}{6}$ exists and $\frac{d^2V}{dx^2}$ at $x = \frac{65}{6}$ is 0.

$\frac{d^2V}{dx^2}$ at $x = \left(\frac{65}{6}\right)^-$ is negative and $\frac{d^2V}{dx^2}$ at

$x = \left(\frac{65}{6}\right)^+$ is positive.

$\therefore$ $x = \frac{65}{6}$ is a point of inflection.

2. (i) $\tan\theta = \frac{5}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{x}\right)$

(ii) $\frac{d\theta}{dx} = \frac{-5}{5^2 + x^2}$

(iii) (a) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$$= \frac{-5}{5^2 + x^2} \times 20$$

$$= \frac{-5}{5^2 + (50)^2} \times 20 \qquad \text{[Given } x = 50 \text{ m]}$$

$$= \frac{-5}{2525} \times 20 = \frac{-4}{101} \text{radian/sec}$$

OR

(b) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{-5}{5^2 + x^2}\bigg|_{x=50} \times \frac{dx}{dt}$

$$\Rightarrow \frac{3}{101} = \frac{-5}{2525} \times \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -15 \text{ m/s}$$

Hence the speed is 15 m/s.

3. (i) When $V = 40$ km/h,

$$F = \frac{(40)^2}{500} - \frac{40}{4} + 14$$

$$= \frac{1600}{500} + 4 = \frac{36}{5} \; (l/100 \text{ km})$$

(ii) $\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}$

(iii) (a) $\frac{dF}{dV} = 0$

$\Rightarrow V = 62.5$ km/h

$$\frac{d^2F}{dV^2} = \frac{1}{250} > 0 \text{ at } V = 62.5 \text{ km/h}$$

Hence, F is minimum when $V = 62.5$ km/h

(b) $\frac{dF}{dV} = -0.01$

$$\Rightarrow \frac{V}{250} - \frac{1}{4} = \frac{-1}{100}$$

$$V = \frac{24}{100} \times 250$$

$\Rightarrow V = 60$ km/h

$$F = \frac{60^2}{500} - \frac{60}{4} + 14 = 6.2 \, l / 100 \text{ km}$$

Quantity of fuel required for 600 km

$= 6.2 \times 6 = 37.2 \, l$

4. (i) Revenue by selling x items =

$$R(x) = x.p(x) = 450x - \frac{x^2}{2}$$

$$\frac{dR}{dx} = 450 - x$$

For Maxima or Minima, $\frac{dR}{dx} = 0 \Rightarrow x = 450$

$$\frac{d^2R}{dx^2} = -1 < 0$$

(Revenue is maximum when $x = 450$ units are sold)

(ii) At $x = 450, p = 450 - \frac{450}{2} = 225$

So, Rebate = 350 – 225 = ₹ 125 per calculator

5. (i) Let $A(x)$ be the area of the visiting card then,

As $xy = 24$,

$$A(x) = (x + 3)(y + 2)$$

$$= 2x + 3y + xy + 6$$

$$= 2x + \frac{72}{x} + 30$$

(ii) $A'(x) = 2 - \frac{72}{x^2}$ and $A''(x) = \frac{144}{x^3}$.

solving $A'(x) = 0 \Rightarrow x = 6$ is the critical point.

$$A''(6) = \frac{144}{(6)^3} > 0,$$

$\therefore$ Area of the card is minimum at $x = 6, y = 4$

The dimension of the card with minimum area is

Length = 9 cm, Breadth = 6 cm

6. (i) No, they are changing at the different rates because they depend on powers of r (like r, r^2 etc.)

(ii) $S = 4\pi r^2, V = \frac{4}{3}\pi r^3$

(iii) (a) $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$$\Rightarrow \frac{dS}{dt}\bigg|_{r=6 \text{ cm}} = 8\pi(6)(2) = 96\pi \text{ cm}^2/\text{s}$$

$$\left[\because \frac{dr}{dt} = 2 \text{ cm/s}\right]$$

OR

(b) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=6\text{ cm}} = 4\pi(6)^2(2) = 288\pi \text{ cm}^3/\text{s}$$

$$\left[\because \frac{dr}{dt} = 2 \text{ cm/s}\right]$$

7. (i) $v = \frac{1}{3}\pi r^3 h = \frac{1}{3}\pi r^3$ [as $\theta = 45°$ gives $r = h$]

(ii) $\frac{dv}{dt} = \pi r^2 \frac{dr}{dt} \Rightarrow -2 = \pi(2\sqrt{2})^2 \frac{dr}{dt}$

$$\left[\because \frac{dv}{dt} = -2 \text{ cm}^3/\text{s and } r = 2\sqrt{2} \text{ cm}\right]$$

$$\Rightarrow \left(\frac{dr}{dt}\right)_{r=2\sqrt{2}} = \frac{-1}{4\pi} \text{ cm/sec}$$

(iii) (a) $C = \pi rl = \pi r\sqrt{2}r = \sqrt{2}\pi r^2$

$$[\because l = \sqrt{r^2+h^2} = \sqrt{r^2+r^2} = \sqrt{2}r]$$

$$\frac{dC}{dt} = \sqrt{2}\pi\, 2r \frac{dr}{dt}$$

$$\left(\frac{dC}{dt}\right)_{r=2\sqrt{2}} = -2 \text{ cm}^2/\text{sec}$$

$$\left[\because \frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/sec}\right]$$

OR

(b) Here, $r = h = 2\sqrt{2}$

$$h = r \Rightarrow \frac{dh}{dt} = \frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/sec}$$

8. (i) Given, $\pi r^2 + 2\pi rh = 75\pi$

$$\Rightarrow h = \frac{75 - r^2}{2r},$$

$$\therefore V = \pi r^2 h = \frac{\pi}{2}(75r - r^3)$$

(ii) $\frac{dV}{dr} = \frac{\pi}{2}(75 - 3r^2)$

(iii) $\frac{dV}{dr} = 0 \Rightarrow r = 5, \left.\frac{d^2V}{dr^2}\right]_{r=5} = \frac{\pi}{2}(-6r) < 0$

$\therefore$ Volume is maximum when $r = 5$

OR

False,

$$\frac{dV}{dr} = 0 \Rightarrow r = 5, \left.\frac{d^2V}{dr^2}\right]_{r=5} = \frac{\pi}{2}(-6r) < 0$$

$\therefore$ Volume is maximum when $r = 5$

As volume is maximum at $r = 5$.

$$\Rightarrow h = \frac{75 - 5^2}{2(5)} = 5 \Rightarrow h = r$$

9. (i) For the year 2000, $t = 0$ & $V(0) = -2$ and the number of vehicles cannot be negative.

$\therefore$ The given function $V(t)$ cannot be used.

(ii) $V'(t) = \frac{3}{5}t^2 - 5t + 25 = \frac{3}{5}\left[\left(t - \frac{25}{6}\right)^2 + \frac{875}{36}\right] > 0,$

$\therefore$ $V(t)$ is an increasing function.

10. (i) $C = 40000\,h^2 + 5000\,x^2$

Given, $x^2h = 250$

$$\Rightarrow C = \frac{40000(250)^2}{x^4} + 5000x^2$$

(ii) $\frac{dC}{dx} = \frac{-160000(250)^2}{x^5} + 10000x$

(iii) (a) For minimum cost $\frac{dC}{dx} = 0$

$\Rightarrow 10000x^6 = 250 \times 250 \times 160000$

$\Rightarrow x = 10$

Also, $\frac{d^2C}{dx^2} > 0$ at $x = 10$

$\therefore$ Cost is minimm when $x = 10$

OR

(b) $\frac{dC}{dx} = \frac{-160000(250)^2}{x^5} + 10000x$

$\frac{dC}{dx} = 0$ gives $x = 10$

$\frac{dC}{dx} > 0$ in $(10, \infty)$ and $\frac{dC}{dx} < 0$ in $(0, 10)$.

Hence, cost function is neither increasing nor decreasing for $x > 0$

11. (i) Capacity = area × depth

$= x^2h = 250$

$$\Rightarrow x^2 = \frac{250}{h}$$

$$C(\text{cost}) = 500\,x^2 + 4000h^2$$

$$\Rightarrow C = 500\left(\frac{250}{h}\right) + 4000h^2$$

$$= \frac{125000}{h} + 4000h^2$$

(ii) $\frac{dC}{dh} = -\frac{125000}{h^2} + 8000h$

$$\frac{dC}{dh} = 0$$

$$\Rightarrow h = \frac{5}{2}\text{m or } 2.5 \text{ m}$$

(iii) (a) $\frac{d^2C}{dh^2} = -125000\left(\frac{-2}{h^3}\right) + 8000$

$$= \frac{250000}{h^3} + 8000$$

$$\left.\frac{d^2C}{dh^2}\right|_{h=2.5\text{m}} > 0$$

$\Rightarrow$ Cost is minimum when $h = 2.5$ m

Minimum cost

$$C = \frac{125000}{\left(\frac{5}{2}\right)} + 4000\left(\frac{5}{2}\right)^2 = ₹\, 75,000$$

OR

(b) We already have found above that $h = \frac{5}{2}m$

when $\frac{dC}{dh} = 0$

For the values of h less than $\frac{5}{2}$ and close to $\frac{5}{2}$,

$\frac{dC}{dh} < 0$

and, for the values of h more than $\frac{5}{2}$ and close to $\frac{5}{2}$, $\frac{dC}{dh} > 0$

By first derivative test, there is a minimum at $h = \frac{5}{2}$

Now, $x^2 = \frac{250}{h} \Rightarrow x^2 = \frac{250}{\left(\frac{5}{2}\right)} = 100 \Rightarrow x = 10$ m

Also, $x = 4h$

12. (i) $f'(x) = x^2 - 8x + 15 = (x-3)(x-5)$

$f'(x) = 0 \Rightarrow x = 3, 5$ are the critical points.

(ii) Now $f''(x) = 2x - 8$

$f''(3) < 0$ and $f''(5) > 0$

So, minimum value of $f(x)$ is at $x = 5$.

$$\text{Minimum value} = f(5) = \frac{5^3}{3} - 4(5)^2 + 15(5) + 2 = \frac{56}{3}$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. (i) The yield (r) per day 3 years after planting the sapling is:

$$r = \frac{3^3}{3} - 6(3^2) + 32(3)$$

$\Rightarrow$ $r = 51$ grams

(ii) The first derivative of the given function with respect to time is:

$$r'(t) = t^2 - 12t + 32$$

Put $r'(t) = 0 \Rightarrow t = 4$ and $t = 8$.

Since the life span of the plant is around 6 years, only $t = 4$ years is applicable here.

The second derivative of the given function with respect to time is:

$$r''(t) = 2t - 12$$

$r''(t)$ at $t = 4$ years is:

$$r''(4) = (-4)$$

By the second derivative test, $t = 4$ years is the point of maxima. Hence, concludes that the maximum yield per day is obtained 4 years after planting the saplings.

2. Assumes the width of photo frame to be x cm and its length to be $\frac{80}{x}$ cm.

Subtracts the specified margins from the width and length to find the area (A) available to stick the photo as:

$$A = (x-2)\left(\frac{80}{x} - \frac{5}{2}\right)$$

$$\Rightarrow \quad A = 85 - \frac{5}{2}x - \frac{160}{x}$$

Differentiates area with respect to x as:

$$\frac{dA}{dx} = -\frac{5}{2} + \frac{160}{x^2}$$

$$\frac{d^2A}{dx^2} = \frac{-320}{x^3}$$

Equates the above derivative to zero and finds the critical point as:

$$-\frac{5}{2} + \frac{160}{x^2} = 0$$

$$\Rightarrow \quad x^2 = 64$$

$\Rightarrow x = 8$ (as x being a length cannot be negative)

Now, $\frac{d^2A}{dx^2}$ at $x = 8$ as:

$$\frac{d^2A}{dx^2} \text{ (at } x = 8) = -\frac{5}{8} < 0$$

Concludes that by second derivative test, the area is maximum at $x = 8$ cm.

Thus, the length of the frame $= \frac{80}{8} = 10$ cm.

Hence the required dimensions of the photo frame are 8 cm and 10 cm.

3. Takes x as the distance between the lantern and the ground, θ as camera's angle of elevation in radians and t as the time in minutes.

Using tangent function, we get

$$\tan\theta = \frac{x}{100} \qquad \text{...(i)}$$

Differentiating the above equation with respect to t, we get:

$$\sec^2\theta \times \frac{d\theta}{dt} = \frac{1}{100} \times \frac{dx}{dt}$$

Now,

$$\frac{d\theta}{dt} = \frac{1}{4\sec^2\theta} \qquad \text{...(ii)}$$

$$\left[\because \text{Given, } \frac{dx}{dt} = 25 \text{ m/min}\right]$$

Let y is the distance between the camera and the lantern.

$$\therefore \sec\theta = \frac{y}{100}$$

Using the Pythagoras theorem to find y as:

$$y^2 = x^2 + 100^2$$

$$\Rightarrow \quad y^2 = 75^2 + 100^2$$

$$\Rightarrow \quad y = 125, \text{ as } y > 0.$$

Substitutes $y = 125$ to get $\sec\theta$ as $\frac{5}{4}$.

Substitutes the value of $\sec\theta$ in the equation (ii), we get $\frac{d\theta}{dt} = \frac{4}{25}$ or 0.16 radians/min.

4. Considers the length of PQ as x cm and finds $\angle$QPR and $\angle$QPS as:

$$\angle QPR = \tan^{-1}\left(\frac{2}{x}\right)$$

$$\angle QPS = \tan^{-1}\left(\frac{8}{x}\right)$$

Considers $\angle$RPS as θ finds θ in terms of x as:

$$\theta = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$$

Finds the derivative of θ with respect to x as follows:

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\left(\frac{8}{x}\right)\right) - \frac{d}{dx}\left(\tan^{-1}\left(\frac{2}{x}\right)\right)$$

Applies chain rule and differentiates as follows:

$$\frac{dQ}{dx} = \frac{1}{\left(\frac{8}{x}\right)^2 + 1} \cdot \frac{d}{dx}\left(\frac{8}{x}\right) - \frac{1}{\left(\frac{2}{x}\right)^2 + 1} \cdot \frac{d}{dx}\left(\frac{2}{x}\right)$$

$$= \frac{2}{\left[\left(\frac{4}{x^2} + 1\right)x^2\right]} - \frac{8}{\left[\left(\frac{64}{x^2} + 1\right)x^2\right]}$$

Equates the derivative to 0 to find that the maximum value of θ will occur when x is either 4 or (–4). States that the minimum value of θ is 0, which cannot occur at $x = 4$ and hence it must be the maxima. The working may look as follows:

$$\frac{8}{\left[\left(\frac{64}{x^2} + 1\right)x^2\right]} = \frac{2}{\left[\left(\frac{4}{x^2} + 1\right)x^2\right]}$$

Cancelling x^2 on both sides,

$$4\left(\frac{4}{x^2} + 1\right) = \frac{64}{x^2} + 1$$

Rearranges terms to obtain,

$$x^2 = 16$$

$\Rightarrow x = +4$ or -4

Ignores $x = -4$ as the length of PQ cannot be negative and writes $\angle$RPS will be maximum when length of PQ = 4 cm.