CHAPTER 7

Integrals

Level - 1 CORE SUBJECTIVE QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ) (1 Marks)

1. **Option (A) is correct.**

   Explanation:

$$\int \frac{dx}{x^3(1+x^4)^{\frac{1}{2}}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{1}{2}}}$$

(Let $1+\frac{1}{x^4}=t$, $dt=-4x^5dx=-\frac{4}{x^5}dx \Rightarrow \frac{dx}{x^5}=-\frac{1}{4}dt$)

$$=-\frac{1}{4}\int\frac{dt}{t^{\frac{1}{2}}}=-\frac{1}{4}\times 2\times\sqrt{t}+c,$$

where '*c*' denotes any arbitrary constant of integration.

$$=-\frac{1}{2}\sqrt{1+\frac{1}{x^4}}+c=-\frac{1}{2x^2}\sqrt{1+x^4}+c$$

2. **Option (A) is correct.**

   Explanation: We know,

$$\int_0^{2a} f(x)dx=0, \text{ if } f(2a-x)=-f(x)$$

Let $f(x) = \text{cosec}^7 x$.

Now, $f(2\pi - x) = \text{cosec}^7(2\pi - x) = -\text{cosec}^7 x = -f(x)$

$$\therefore \int_0^{2\pi}\text{cosec}^7 x\,dx=0;$$

3. **Option (D) is correct.**

   Explanation: We have,

$$\int\frac{\cos 2x}{\sin^2 x\cos^2 x}dx = \int\frac{\cos^2 x-\sin^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\frac{\cos^2 x}{\sin^2 x\cos^2 x}dx-\int\frac{\sin^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\frac{1}{\sin^2 x}dx-\int\frac{1}{\cos^2 x}dx$$

$$=\int\text{cosec}^2 x\,dx-\int\sec^2 x\,dx$$

$$=-\cot x-\tan x+c$$

4. **Option (B) is correct.**

   Explanation: Let

$$I=\int_a^b f(x)dx$$

Put $x = a + b - t$

$\therefore$ $dx = -dt$

Now, when $x = a$, $t = b$

and when $x = b$, $t = a$

$$\therefore \quad I=\int_b^a f(a+b-t)(-dt)$$

$$I=-\int_b^a f(a+b-t)dt$$

$$=\int_a^b f(a+b-t)dt$$

$$\left[\because \int_a^b f(x)dx=-\int_b^a f(x)dx\right]$$

Now, replacing $t \to x$

$$\therefore \quad I=\int_a^b f(a+b-x)dx$$

5. **Option (B) is correct.**

   Explanation: $\int_0^a \frac{1}{4+x^2}dx=\frac{\pi}{6}$

Using the standard formula:

$$\int\frac{1}{p^2+x^2}dx=\frac{1}{p}\tan^{-1}\frac{x}{p}$$

Here, $p^2 = 4$ so $p = 2$:

$$\int_0^a\frac{1}{4+x^2}dx=\frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_0^a$$

$$\frac{1}{2}\left(\tan^{-1}\frac{a}{2}-\tan^{-1}0\right)=\frac{\pi}{6}$$

$$\frac{1}{2}\tan^{-1}\frac{a}{2}=\frac{\pi}{6}$$

$$\tan^{-1}\frac{a}{2}=\frac{\pi}{3}$$

$$\frac{a}{2}=\tan\frac{\pi}{3}=\sqrt{3}$$

$$a=2\sqrt{3}$$

6. Option (B) is correct.

Explanation: $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos^3 x dx$

Since $f(x)$ is odd, it satisfies:

$$f(-x) = -f(x)$$

Also, cosine is even, meaning:

$$\cos^3(-x) = \cos^3 x$$

Thus, substituting $x \to -x$ in the integral:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(-x)\cos^3(-x)dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-f(x))\cos^3 x dx$$

$$= -I$$

We get:

$$2I = 0 \Rightarrow I = 0$$

7. Option (B) is correct.

Explanation:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}dx$

Let $f(x) = \frac{\sin x - \cos x}{1+\sin x\cos x}$

Using the transformation $x \to \frac{\pi}{2} - x$

$$f\left(\frac{\pi}{2} - x\right) = \frac{\cos x - \sin x}{1+\sin x \cos x} = -f(x)$$

Thus,

$$I = \int_0^{\frac{\pi}{2}} -f(x)dx = -I$$

$$2I = 0 \Rightarrow I = 0$$

8. Option (C) is correct.

Explanation: LHS = $\int 2e^{2x}dx$

Using $\int e^{ax}dx = \frac{e^{ax}}{a}$

$$\int 2e^{2x}dx = 2\cdot\frac{e^{2x}}{2} = e^{2x}$$

Evaluating from 0 to 2 :

$$e^4 - e^0 = e^4 - 1$$

Right-Hand Side Integral

$$\int_0^a e^x dx = e^x\Big|_0^a = e^a - 1$$

Equating Both Sides

$$e^4 - 1 = e^a - 1$$

On comparing

$$a = 4$$

9. Option (B) is correct.

Explanation: Let $I = \int\sqrt{1+\sin 2x}dx$

We know that,

$$1 + \sin 2x = (\cos x + \sin x)^2$$

$\therefore$ $I = \int\sqrt{(\cos x+\sin x)^2}dx$

$$I = \int (\cos x + \sin x)dx$$

$$I = \int\cos x dx + \int\sin x dx$$

$$I = \sin x - \cos x + c$$

10. Option (B) is correct.

Explanation: Condition for $\int_{-a}^{a} f(x)dx = 0$

The property of definite integrals states:

$\int_{-a}^{a} f(x)dx = 0$, if $f(x)$ is odd, i.e., $f(-x) = -f(x)$

11. Option (C) is correct.

Explanation:

Using the standard integral:

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

Here, $a = 3$, so:

$$\int_0^3\frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

12. Option (C) is correct.

Explanation: Since $|x|$ is defined as:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

We split the integral:

$$\int_{-1}^{1}|x|dx = \int_{-1}^{0}(-x)dx + \int_0^1 xdx$$

Now, $\int_{-1}^{0} -xdx = -\frac{x^2}{2}\Big|_{-1}^{0} = -\left(0 - \frac{1}{2}\right) = \frac{1}{2}$

$$\int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$\therefore$ $\int_{-1}^{1}|x|dx = \frac{1}{2} + \frac{1}{2} = 1$

13. Option (B) is correct.

Explanation: Let $I = \int_0^{\pi}\tan^2\left(\frac{\theta}{3}\right)d\theta$

Using $\tan^2 x = \sec^2 x - 1$,

$$I = \int_0^{\pi}\left(\sec^2\left(\frac{\theta}{3}\right) - 1\right)d\theta$$

Splitting,

$$I = \int_0^{\pi}\sec^2\left(\frac{\theta}{3}\right)d\theta - \pi$$

Let $$I_1 = \int_0^{\pi}\sec^2\left(\frac{\theta}{3}\right)d\theta$$

Substituting $x = \frac{\theta}{3}, d\theta = 3dx$

$$\therefore \quad I_1 = 3\int_0^{\pi/3}\sec^2 x dx = 3[\tan x]_0^{\pi/3}$$

$$= 3(\sqrt{3} - 0) = 3\sqrt{3}$$

Thus, $$I = 3\sqrt{3} - \pi$$

14. Option (B) is correct.

Explanation:

Let $u = \log x$, so $du = \frac{dx}{x}$, transforming the integral into:

$$\int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\log x} + c$$

15. Option (D) is correct.

Explanation:

Here, $$x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

Now, integral

$$\int_{-1}^{1} x|x|dx = \int_{-1}^{0} -x^2 dx + \int_0^1 x^2 dx$$

$$\int_{-1}^{0} -x^2 dx = -\left.\frac{x^3}{3}\right|_{-1}^{0} = -\left(0 - \left(-\frac{1}{3}\right)\right)$$

$$= -\frac{1}{3}$$

$$\int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$\therefore \quad \int_{-1}^{1} x|x|dx = -\frac{1}{3} + \frac{1}{3} = 0$$

16. Option (A) is correct.

Explanation: Let

$$I = \int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta d\theta$$

Put $t = \cot\theta$, so $dt = -csc^2\theta d\theta$,

$$I = \int_{\pi/4}^{\pi/2} t(-dt) = -\int_1^0 t dt$$

$$= -\left[\frac{t^2}{2}\right]_1^0 = -\left(0 - \frac{1}{2}\right) = \frac{1}{2}$$

17. Option (B) is correct.

Explanation: The integral is of the form:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Comparing with $\sqrt{9 - 4x^2}$, we rewrite:

$$9 - 4x^2 = 3^2 - (2x)^2$$

$$\therefore \quad \int \frac{dx}{\sqrt{3^2 - (2x)^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$$

18. Option (A) is correct.

Explanation:

We have, $\int_{-2}^{3} x^2 dx = k\int_0^2 x^2 dx + \int_2^3 x^2 dx$

Evaluating Each Integral

We compute:

$$\int x^2 dx = \frac{x^3}{3}$$

Computing $\int_{-2}^{3} x^2 dx$

$$\left[\frac{x^3}{3}\right]_{-2}^{3} = \left[\frac{3^3}{3}\right] - \left[\frac{(-2)^3}{3}\right]$$

$$= 9 + \frac{8}{3}$$

$$= \frac{35}{3}$$

Computing $\int_0^2 x^2 dx$

$$\left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

Computing $\int_2^3 x^2 dx$

$$\left[\frac{x^3}{3}\right]_2^3 = \frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

Solving for k

$$\frac{35}{3} = k\cdot\frac{8}{3}+\frac{19}{3}$$

$$35 = 8k + 19$$

$$8k = 16 \Rightarrow k = 2$$

19. Option (B) is correct.

Explanation: $\int_1^e \log x dx$

Using integration by parts, let:

$$u = \log x,\ \text{so}\ du = \frac{dx}{x}$$

$dv = dx$, so $v = x$.

Applying the formula:

$$\int u dv = uv - \int v du$$

$$\int \log x dx = x\log x - x$$

Evaluating from 1 to e:

$$[e\log e - e] - [1\log 1 - 1]$$

Since $\log e = 1$ and $\log 1 = 0$, this simplifies to:

$$[e - e] - [0 - 1] = 1$$

20. Option (A) is correct.

Explanation: Since x^3 is an odd function and $\cos^2 x$ is even, their product remains odd.

For any odd function, integration over symmetric limits gives zero.

21. Option (C) is correct.

Explanation: Using property:

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Let $f(x) = \log(\tan x)$. Substituting $x \to \frac{\pi}{2} - x$, we get:

$$\log\left(\tan\left(\frac{\pi}{2}-x\right)\right) = \log(\cot x) = -\log(\tan x)$$

Thus,

$$I = \int_{\pi/6}^{\pi/3} \log(\tan x)dx = -I$$

$$2I = 0 \Rightarrow I = 0$$

22. Option (C) is correct.

Explanation:

$$\int \frac{dx}{\sin^2 2x\cos^2 2x} = \int \frac{1}{\sin^2 2x\cos^2 2x}\cdot dx$$

$$= \int \frac{\sin^2 2x + \cos^2 2x}{\sin^2 2x\cos^2 2x}\cdot dx$$

(Using $\sin^2 x + \cos^2 x = 1$)

$$= \int \frac{\sin^2 2x}{\sin^2 2x\cos^2 2x}\cdot dx + \int \frac{\cos^2 2x}{\sin^2 2x\cos^2 2x}\cdot dx$$

$$= \int \frac{1}{\cos^2 2x}\cdot dx + \int \frac{1}{\sin^2 2x}\cdot dx$$

$$= \int \sec^2 2x \cdot dx + \int \text{cosec}^2 2x\cdot dx$$

$$= \frac{1}{2}\tan 2x - \frac{1}{2}\cot 2x + C$$

$$= \frac{1}{2}[\tan 2x - \cot 2x] + C$$

23. Option (B) is correct.

Explanation: $I = \int_{-\pi/4}^{\pi/4} \sin^3 x dx$

Since $\sin^3 x$ is an odd function ($\sin(-x) = -\sin x$), we use the property:

$$\int_{-a}^{a} f(x)dx = 0,\ \text{if } f(x) \text{ is odd}$$

Hence, $\sin^3 x$ is odd, the integral evaluates to 0.

24. Option (A) is correct.

Explanation: $I = \int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right)dx$

We know:

$$\int \sec^2 u du = \tan u + C$$

Using $u = x - \frac{\pi}{6}$, we get:

$$I = \left[\tan\left(x - \frac{\pi}{6}\right)\right]_0^{\pi/6}$$

$$I = \tan\left(\frac{\pi}{6} - \frac{\pi}{6}\right) - \tan\left(0 - \frac{\pi}{6}\right)$$

$$= \tan(0) - \tan\left(\frac{-\pi}{6}\right)$$

$$= 0 - \left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

25. Option (B) is correct.

Explanation: Given:

$$\frac{d}{dx}f(x) = ax + b,\ f(0) = 0$$

Integrate

$$f(x) = \int (ax+b)dx$$

$$= \frac{ax^2}{2} + bx + C$$

Use $f(0) = 0$

$$0 = \frac{a(0)^2}{2} + b(0) + C$$

$$C = 0$$

$$f(x) = \frac{ax^2}{2} + bx$$

26. Option (C) is correct.

Explanation:

$$\int 2^{x+2}dx = \int 2^2 \cdot 2^x dx$$

$$= 4\int 2^x dx$$

Using formula

$$\int a^x dx = \frac{a^x}{\ln a}$$

For $a = 2$, we get:

$$I = 4\cdot\frac{2^x}{\ln 2} + C$$

$$= \frac{2^{x+2}}{\ln 2} + C$$

27. Option (B) is correct.

Explanation: Using the property:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Substituting $x \to \frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right)dx$$

Since $\tan\left(\frac{\pi}{2} - x\right) = \cot x$, we get:

$$I = \int_0^{\pi/2} \log \cot x\, dx$$

Using the identity:

$$\log \tan x + \log \cot x = 0$$

$$I + I = 0 \Rightarrow 2I = 0 \Rightarrow I = 0$$

28. Option (B) is correct.

Explanation: $I = \int e^{5\log x}dx$

Using the identity:

$$e^{\log_a b} = a^b$$

$$e^{5 \log x} = x^5$$

Thus,

$$I = \int x^5 dx = \frac{x^6}{6} + C$$

29. Option (A) is correct.

Explanation: $\int_0^a 3x^2 dx = 8$

Solve the integral

$$\int 3x^2 dx = 3\frac{x^3}{3} = x^3$$

Evaluating from 0 to a:

$$a^3 - 0 = 8$$

$$a^3 = 8$$

$$a = 2$$

30. Option (A) is correct.

Explanation: $\int_0^4 \left(e^{2x} + x\right)dx$

Solving separately:

$$\int e^{2x}dx = \frac{e^{2x}}{2}$$

$$\int x dx = \frac{x^2}{2}$$

Evaluating from 0 to 4,

$$\frac{e^8}{2} - \frac{1}{2} + \frac{16}{2} - 0 = \frac{e^8 + 15}{2}$$

ASSERTION-REASON QUESTIONS (1 Marks)

1. Option (A) is correct.

Explanation:

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}dx$$

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x} + \sqrt{x}}dx$$

By properly,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Now, adding both integrals:

$$I + I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}dx + \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x} + \sqrt{x}}dx$$

$$= \int_2^8 1dx$$

$$= [x]_2^8 = 8 - 2 = 6$$

Thus,

$$2I = 6 \Rightarrow I = 3$$

So the given assertion (A) is correct.

This property was used in our solution to evaluate the integral. Since reason correctly helped solve the problem, the reason (R) is also correct and correctly explains Assertion (A).

2. Option (A) is correct.

Explanation: We have,

$$I = \int x e^{x^2} dx$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$\therefore \quad I = \frac{1}{2}\int e^t\, dt$$

$$= \frac{1}{2}e^t + c$$

$$= \frac{1}{2}e^{x^2} + c$$

3. Option (A) is correct.

***Explanation*:** We know that, differentiation is the inverse process of integration.

Thus, $\frac{d}{dx}\left(\frac{3^x}{\log_e 3}\right) = \frac{1}{\log_e 3}\frac{d}{dx}(3^x)$

$= \frac{1}{\log_e 3}3^x \log_e 3$

$= 3^x$

4. Option (A) is correct.

***Explanation*:** Let, $I = \int 3x^2(\cos x^3 + 8)dx$

Put $x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$\therefore I = \int(\cos t + 8)dt$

$= \int \cos t\, dt + 8\int dt$

$= \sin t + 8t + C$

Thus, $I = \sin x^3 + 8x^3 + C$

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. Let $I = \int x\sqrt{1+2x}\, dx$

Put $1 + 2x = t^2$

$2dx = 2t\, dt$

$\therefore I = \frac{1}{2}\int(t^4 - t^2)dt = \frac{1}{2}\left|\frac{t^5}{5} - \frac{t^3}{3}\right| + C$

$= \frac{1}{30}t^3(3t^2 - 5) + C$

$= \frac{1}{30}(1+2x)^{3/2}(6x-2) + C$

$= \frac{1}{15}(1+2x)^{3/2}(3x-1) + C$

2. Let $I = \int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Put $\sqrt{x} = t \Rightarrow dx = 2t\,dt$

$\therefore I = 2\int_0^{\frac{\pi}{2}} \sin t\, dt = 2[-\cos t]_0^{\frac{\pi}{2}}$

$= 2$

3. Let, $I = \int \frac{e^{4x}-1}{e^{4x}+1}dx$

$= \int \frac{e^{2x} - e^{-2x}}{e^{2x}+e^{-2x}}dx$

$= \frac{1}{2}\int \frac{2(e^{2x}-e^{-2x})}{e^{2x}+e^{-2x}}dx$

$= \frac{1}{2}\log|e^{2x}+e^{-2x}| + C$

4. Put $x^3 = t \Rightarrow x^2 dx = \frac{dt}{3}$

Rewriting the given interal as:

$\frac{1}{3}\int_0^{a^9}\frac{dt}{t^2+(a^3)^2} = \frac{1}{3a^3}\tan^{-1}\frac{t}{a^3}\Big|_0^{a^9}$

$= \frac{1}{3a^3}\tan^{-1}a^6$

5. Let, $I = \int_0^{\pi/2}\sin 2x\cos 3x\,dx$

$= \frac{1}{2}\int_0^{\pi/2}(\sin 5x - \sin x)dx$

$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right]_0^{\frac{\pi}{2}}$

$= -\frac{2}{5}$

6. $F(x) = \int \frac{1}{\sqrt{2x-x^2}}dx$

$= \int \frac{1}{\sqrt{1-(x-1)^2}}dx$

$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$

$= \sin^{-1}(x-1) + c$

when $x = 1$, $F(1) = 0$ gives $c = 0$

$\therefore F(x) = \sin^{-1}(x-1)$

7. Let $I = \int \frac{2x}{(x^2+1)(x^2-4)}dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt$

$I = \int \frac{1}{(t+1)(t-4)}dt$

$= \frac{1}{5}\int\frac{dt}{t-4} - \frac{1}{5}\int\frac{dt}{t+1}$

$= \frac{1}{5}\log|t-4|$

$= \frac{1}{5}\log|t+1| + C$

$\therefore I = \frac{1}{5}\log|x^2-4| - \frac{1}{5}\log|x^2+1|$

$+ c$ or $\frac{1}{5}\log\left|\frac{x^2-4}{x^2+1}\right| + c$

8. Let $f(x) = \cos x . \log\left(\frac{1-x}{1+x}\right)$

So, $f(-x) = \cos(x).\log\left(\frac{1+x}{1-x}\right) = -f(x)$ [Odd function]

Thus, $I = \int_{-1/2}^{1/2} \cos x . \log\left(\frac{1-x}{1+x}\right) dx = 0$

9. Let
$$I = \int \frac{dx}{x(x^2-1)}$$
$$= \int \frac{dx}{x^3\left(1-\frac{1}{x^2}\right)}$$
$$= \frac{1}{2}\int \frac{\left(\frac{2}{x^3}\right)dx}{\left(1-\frac{1}{x^2}\right)}$$

Put $\left(1-\frac{1}{x^2}\right) = t \Rightarrow \left(\frac{2}{x^3}\right)dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + c = \frac{1}{2}\log\left|\left(1-\frac{1}{x^2}\right)\right| + c$$
$$= \frac{1}{2}\log\left|\left(\frac{x^2-1}{x^2}\right)\right| + c$$

10. $\int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x . \sin x dx,$

Assuming $\cos x = t$ and $\sin x\, dx = -dt$
$$= -\int t^3 dt$$
$$= -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

11. $\int \frac{1}{5+4x-x^2} dx = \int \frac{1}{3^2-(x-2)^2} dx$
$$= \frac{1}{6}\log\left|\frac{1+x}{5-x}\right| + C$$
$$\left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

12. $\int \frac{x^3-1}{x^3-x} dx = \int \left(1+\frac{1}{x}-\frac{1}{x+1}\right)dx$
$$= x + \log|x| - \log|x+1| + c$$

13. $\int_{-4}^{0} |x+2|\, dx = \int_{-4}^{-2} -(x+2)dx + \int_{-2}^{0}(x+2)\, dx$
$$= -\left[\frac{(x+2)^2}{2}\right]_{-4}^{-2} + \left[\frac{(x+2)^2}{2}\right]_{-2}^{0}$$
$$= 2 + 2 = 4$$

14.
$$F(x) = \int \tan^4 x\, dx$$
$$= \int \tan^2 x.(\sec^2 x - 1)dx$$
$$= \int (\tan^2 x \sec^2 x - \sec^2 x + 1)dx$$
$[\because \tan^2 x = \sec^2 x - 1]$
$$F(x) = \frac{\tan^3 x}{3} - \tan x + x + C$$
$x = \frac{\pi}{4}$, $F(x) = \frac{\pi}{4}$ gives $C = \frac{2}{3}$
$$F(x) = \frac{\tan^3 x}{3} - \tan x + x + \frac{2}{3}$$

15. Put $x = e^t$ or $\log x = t$ $\therefore$ $dx = e^t dt$
$$\therefore \int \frac{\log x - 3}{(\log x)^4} dx = \int \frac{t-3}{t^4} e^t dt$$
$$= \int \left(\frac{1}{t^3} - \frac{3}{t^4}\right) e^t dt$$
$$= \frac{e^t}{t^3} + C$$
$[\because \int (f(x) + f(x)]\, e^x dx = e^x f(x) + c]$
$$= \frac{x}{(\log x)^3} + C$$

16.
$$\int \frac{\sin 3x}{\sin x} dx = \int \frac{3\sin x - 4\sin^3 x}{\sin x}$$
$$= \int \left[3 - 4\frac{(1-\cos 2x)}{2}\right] dx$$
$$= \int (1 + 2\cos 2x)dx$$
$$= x + \sin 2x + C$$

17. Let $e^x = t$, $e^x\, dx = dt$
$$\therefore \int_0^{\frac{1}{2}\log 3} \frac{e^x}{e^{2x}+1} dx = \int_1^{\sqrt{3}} \frac{dt}{t^2+1}$$
$$= \tan^{-1} t\Big]_1^{\sqrt{3}}$$
$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

18. Let $\cot x = t$, then $-\text{cosec}^2 x dx = dt$
$$\therefore \int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = \int \frac{\sqrt{\cot x}}{\cot x} \text{cosec}^2 x dx$$
$$= -\int \frac{\sqrt{t}}{t} dt = -\int \frac{1}{\sqrt{t}} dt$$
$$= -2\sqrt{t} + C$$
$$= -2\sqrt{\cot x} + C$$

19. Let
$$I = \int \frac{1}{x(x^2+4)} dx$$
$$= \int \frac{1}{x^3\left(1+\frac{4}{x^2}\right)} dx$$

Let $1 + \frac{4}{x^2} = t \Rightarrow \frac{-8}{x^3} dx = dt$
$$\therefore \quad I = \frac{-1}{8}\int \frac{dt}{t}$$

$$= \frac{-1}{8}\log|t| + C$$

$$= \frac{-1}{8}\log\left|1 + \frac{4}{x^2}\right| + C$$

$$= \frac{-1}{8}\log\left|\frac{x^2+4}{x^2}\right| + C$$

20. $$\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dx}{(x^2 - 6x + 9) + 4}$$

$$= \int \frac{dx}{(x-3)^2 + (2)^2}$$

$$= \frac{1}{2}\tan^{-1}\frac{(x-3)}{2} + C$$

21. $$\int_0^1 x^2 e^x dx = [x^2e^x]_0^1 - \int_0^1 2xe^x\, dx$$

$$= [x^2e^x - 2xe^x + 2e^x]_0^1$$

$$= e - 2$$

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. $$\int\left\{\frac{1}{(\log_e x)} - \frac{1}{(\log_e x)^2}\right\}dx$$

$$= \int \frac{dx}{\log_e x} - \int \frac{1}{(\log_e x)^2}\,dx$$

$$= \frac{1}{\log_e x}\int dx - \int\left\{\frac{d}{dx}\left(\frac{1}{\log_e x}\right)\int dx\right\}dx - \int \frac{1}{(\log_e x)^2}dx$$

$$= \frac{x}{\log_e x} + \int \frac{1}{(\log_e x)^2}\frac{1}{x}.x.dx - \int \frac{1}{(\log_e x)^2}\,dx$$

$$= \frac{x}{\log_e x} + \int \frac{1}{(\log_e x)^2}\,dx - \int \frac{1}{(\log_e x)^2}\,dx = \frac{x}{\log_e x} + c;$$

where 'c' is any arbitrary constant of integration.

2. $$\int_0^1 x(1-x)^n dx = \int_0^1 (1-x)\{1-(1-x)\}^n dx,$$

$$\left[\text{as, } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 x^n(1-x)dx$$

$$= \int_0^1 x^n dx - \int_0^1 x^{n+1}dx$$

$$= \frac{1}{n+1}[x^{n+1}]_0^1 - \frac{1}{n+2}[x^{n+2}]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

3. Let $$I = \int \frac{x^2}{(x^2+4)(x^2+9)}dx$$

Put $x^2 = t$

$$\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9}$$

$$\Rightarrow A = \frac{-4}{5}, B = \frac{9}{5}$$

$$I = \frac{-4}{5}\int \frac{1}{2^2 + x^2}\,dx + \frac{9}{5}\int \frac{1}{3^2 + x^2}dx$$

$$= \frac{-2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C$$

4. $$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx$$

$$= \int_1^3 (x-1)dx + \int_1^2 -(x-2)dx + \int_2^3 (x-2)dx - \int_1^3 (x-3)dx$$

$$= \int_1^3 2\,dx + \int_1^2 (2-x)dx + \int_2^3 (x-2)dx$$

$$= |2x|_1^3 + \left|\frac{(2-x)^2}{-2}\right|_1^2 + \left|\frac{(x-2)^2}{2}\right|_2^3$$

$$= 4 + \frac{1}{2} + \frac{1}{2} = 5$$

5. $$\int_{-2}^2 \sqrt{\frac{2-x}{2+x}}\,dx = \int_{-2}^2 \frac{2-x}{\sqrt{4-x^2}}\,dx$$

$$= \int_{-2}^2 \frac{2}{\sqrt{4-x^2}}\,dx - \int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\,dx$$

$$= 2\int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx - 0$$

$$\left[\text{Since, } \frac{2}{\sqrt{4-x^2}} \text{ is even, } \frac{x}{\sqrt{4-x^2}} \text{ is odd}\right]$$

$$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= 4\sin^{-1}\frac{x}{2}\Big]_0^2$$

$$= 2\pi$$

6. Let $\log x = t \Rightarrow \frac{1}{x}dx = dt$

The given integral becomes $I = \int \frac{1}{t^2 - 3t - 4}\,dt$

$$= \int \frac{1}{\left(t - \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}\,dx$$

$$= \frac{1}{5}\log\left|\frac{t-4}{t+1}\right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{5}\log\left|\frac{\log x - 4}{\log x + 1}\right| + C$$

7. Let $x^{\frac{3}{2}} = t$

$$\Rightarrow \frac{3}{2}x^{\frac{1}{2}}dx = dt$$

The given integral becomes: $\frac{2}{3}\int t\sin^{-1} t dt$

$$= \frac{2}{3}\left[\sin^{-1} t \times \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \times \frac{t^2}{2} dt\right]$$

$$= \frac{1}{3}\left[\sin^{-1} t \times t^2 + \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt\right]$$

$$= \frac{1}{3}\left[\sin^{-1} t \times t^2 + \int \sqrt{1-t^2}dt - \int \frac{1}{\sqrt{1-t^2}} dt\right]$$

$$= \frac{1}{3}\left[t^2 \sin^{-1} t + \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t - \sin^{-1} t\right] + C$$

$$= \frac{1}{3}\left[t^2 \sin^{-1} t + \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t\right] + C$$

$$= \frac{1}{3}\left[x^3 \sin^{-1}(x)^{\frac{3}{2}} + \frac{x^{\frac{3}{2}}}{2}\sqrt{1-x^3} - \frac{1}{2}\sin^{-1}(x)^{\frac{3}{2}}\right] + C$$

8. Integrating by parts, we get

$$\log(x^2-1) \times \frac{x^3}{3} - \int \frac{2x}{x^2-1} \times \frac{x^3}{3} dx$$

$$= \log(x^2-1) \times \frac{x^3}{3} - \frac{2}{3}\int \frac{x^4-1+1}{x^2-1} dx$$

$$= \log(x^2-1) \times \frac{x^3}{3} - \frac{2}{3}\left[\int (x^2+1)dx + \int \frac{1}{x^2-1} dx\right]$$

$$= \frac{x^3 \log(x^2-1)}{3} - \frac{2}{3}\left[\frac{x^3}{3} + x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right] + C$$

9. $$I = \int \frac{x^2+1}{(x^2+2)(x^2+4)} dx$$

Let $x^2 = y$, then $\frac{x^2+1}{(x^2+2)(x^2+4)} = \frac{y+1}{(y+2)(y+4)}$

Let $\frac{y+1}{(y+2)(y+4)} = \frac{A}{y+2} + \frac{B}{y+4}$

this gives $A = -\frac{1}{2}, B = \frac{3}{2}$

$$\therefore I = -\frac{1}{2}\int \frac{1}{x^2+2} dx + \frac{3}{2}\int \frac{1}{x^2+4} dx$$

$$\Rightarrow I = -\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{3}{4}\tan^{-1}\left(\frac{x}{2}\right) + c$$

10. $$I = \int \frac{2+\sin 2x}{1+\cos 2x} e^x dx$$

$$= \int \frac{2+2\sin x \cos x}{2\cos^2 x} e^x dx$$

$$= \int (\sec^2 x + \tan x)e^x dx$$

$$= e^x.\tan x + c$$

$$\left[\because \int [f(x) + f'(x)e^x dx = e^x f(x) + c\right]$$

11. $$I = \int_0^{\pi/4} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x} dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sin\left(x+\frac{\pi}{4}\right)} dx$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\operatorname{cosec}\left(x+\frac{\pi}{4}\right) - \cot\left(x+\frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1) \text{ or } \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

12. Let $\log x = t$; $\frac{1}{x} dx = dt$

Given integral $= \int \frac{1}{t^2 - 5t + 4} dt$

$$= \int \frac{1}{\left(t-\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dt$$

$$= \frac{1}{3}\log\left|\frac{t-4}{t-1}\right| + C$$

$$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{3}\log\left|\frac{\log x - 4}{\log x - 1}\right| + C$$

13. Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ ...(i)

$$I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} dx$$

$$= \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx \quad \text{...(ii)}$$

$[\because \cos(\pi - x) = -\cos x]$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} dx = x]_0^{x} = \pi, \therefore I = \frac{\pi}{2}$$

14. $\int \frac{2x+1}{(x+1)^2(x-1)}dx$

$$= -\frac{3}{4}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}dx + \frac{3}{4}\int \frac{1}{x-1}dx$$

[By partial fraction]

$$= -\frac{3}{4}\log|x+1| - \frac{1}{2(x+1)} + \frac{3}{4}\log|x-1| + C$$

$$= \frac{3}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2(x+1)} + C$$

15. $\int \frac{dx}{\cos x\sqrt{\cos 2x}} = \int \frac{dx}{\cos x\sqrt{\cos^2 x - \sin^2 x}}$

$$= \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$$

using $\tan x = t$, $\sec^2 x.dx = dt$

$$= \int \frac{1}{\sqrt{1-t^2}}dt,$$

$$= \sin^{-1}t + c = \sin^{-1}(\tan x) + c$$

16. $\int \frac{5x-3}{\sqrt{1+4x-2x^2}}dx$

$$= -\frac{5}{4}\int \frac{-4x+4}{\sqrt{1+4x-2x^2}}dx + \sqrt{2}\int \frac{1}{\sqrt{\left(\left(\sqrt{\frac{3}{2}}\right)^2 - (x-1)^2\right)}}dx$$

$$= -\frac{5}{2}\sqrt{1+4x-2x^2} + \sqrt{2}\sin^{-1}\left(\sqrt{\frac{2}{3}}(x-1)\right) + c$$

17. $I = \int \frac{dx}{1+\cot x} = \int \frac{\sin x}{\sin x + \cos x}dx$

$$= \frac{1}{2}\int \frac{(\sin x+\cos x)-(\cos x-\sin x)}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\int 1.dx - \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx$$

$$= \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C$$

18. $I = \int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$= \int_{\pi/2}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int_{\pi/2}^{\pi} e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= -\int_{\pi/2}^{\pi} e^x\left(\cot\frac{x}{2} - \frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$$

$$\left[\because \int e^x(f(x)+f'(x))dx = e^x f(x) + c\right]$$

$$= -\left[e^x\cot\frac{x}{2}\right]_{\pi/2}^{\pi}$$

$$= e^{\frac{\pi}{2}}$$

19. $I = \int_0^{\pi/4}\log(1+\tan x)\,dx$

$$= \int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$= \int_0^{\pi/4}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4}\log 2dx - \int_0^{\pi/4}\log(1+\tan x)dx$$

$$= \log 2[x]_0^{\pi/4} - I$$

$$2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

20. $I = \int \frac{x}{(x^2+1)(x-1)}dx$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow x = A(x^2+1) + (Bx+C)(x-1)$

$\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}, C = \frac{1}{2}$

$$I = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

21. Let $I = \int e^x \sin x\,dx$

$$= e^x\sin x - \int \cos x\, e^x\,dx$$

$$= e^x\sin x - \cos x\, e^x - I$$

$$\therefore \quad I = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\therefore \int_0^{\pi/2} e^x\sin x\,dx = \frac{1}{2}e^{\pi/2} + \frac{1}{2} \text{ or } \frac{1}{2}(e^{\pi/2}+1)$$

22. Let $I = \int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\left[\int \frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[\int [\tan(x-b) - \tan(x-a)]dx\right]$$

$$= \frac{1}{\sin(a-b)}[\log|\sec(x-b)| - \log|\sec(x-a)|] + C$$

23. Let $I = \int_0^{\pi/2}[\log\sin x - \log(2\cos x)]\,dx$

$$= \int_0^{\pi/2}\log\left(\frac{\tan x}{2}\right)dx$$

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

We get, $I = \int_0^{\pi/2}\log\left(\frac{\cot x}{2}\right)dx$

$\therefore$ $2I = \int_0^{\pi/2}\log\left(\frac{\tan x}{2}\times\frac{\cot x}{2}\right)dx$

$$= \int_0^{\pi/2}\log\left(\frac{1}{4}\right)dx$$

$$2I = \log\left(\frac{1}{4}\right)x\Big|_0^{\pi/2} = \frac{\pi}{2}\log\frac{1}{4}$$

$I = \frac{\pi}{4}\log\frac{1}{4}$ or $-\frac{\pi}{2}\log 2$

24. Let $I = \frac{dx}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)}$

Let $\sqrt{x} = t, \frac{1}{2\sqrt{x}}dx = dt$

$\therefore$ $I = 2\int\frac{dt}{(t+1)(t+2)}$

$$= 2\int\left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$

$= 2\,[\log|t+1| - \log|t+2| + C$

$$= 2[\log(\sqrt{x}+1) - \log(\sqrt{x}+2)] + C$$

or $2\log\left(\frac{\sqrt{x}+1}{\sqrt{x}+2}\right) + C$

25. Let $I = \int_0^{\pi/2}(\sin x - \cos x)\,dx$

$$= \int_0^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2}(\sin x - \cos x)\,dx$$

$$= (\cos x + \sin x)]_0^{\pi/4} + (-\cos x - \sin x)]_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2}-1) - 1 + \sqrt{2}$$

$$= 2\sqrt{2} - 2$$

26. Let $I = \int_{\log\sqrt{2}}^{\log\sqrt{3}}\frac{1}{(e^x+e^{-x})(e^x-e^{-x})}dx$

$$= \int_{\log\sqrt{2}}^{\log\sqrt{3}}\frac{e^{2x}}{(e^{2x})^2-1}dx$$

Put $e^{2x} = t \Rightarrow e^{2x}dx = \frac{1}{2}dt$, Upper limit = 3, Lower limit = 2

$$\therefore I = \frac{1}{2}\int_2^3\frac{1}{t^2-1}dt = \frac{1}{4}\log\left|\frac{t-1}{t+1}\right|\Big]_2^3$$

$$\Rightarrow I = \frac{1}{4}\left[\log\frac{2}{4} - \log\frac{1}{3}\right] = \frac{1}{4}\log\frac{3}{2}$$

27. $I = \int_{-1}^{1}|x^4 - x|\,dx$

$$= \int_{-1}^{0}(x^4-x)dx - \int_0^1(x^4-x)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Big]_{-1}^{0} - \left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Big]_0^1$$

$$= \frac{7}{10} + \frac{3}{10} = 1$$

28. Let $I = \int\frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$

(Putting $\sin^{-1}x = t$, $x = \sin t$, also $\frac{1}{\sqrt{1-x^2}}dx = dt$)

$\therefore$ $I = \int\frac{t\,dt}{(1-x^2)}$

$$= \int\frac{t\,dt}{\cos^2 t}$$

$$= \int t\sec^2 dt$$

On integrating by parts, we get

$$I = t\cdot\tan t + \log|\cos t| + c$$

$$= \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| + c$$

29. $I = \int\frac{\cos x}{3\sin x - 4\sin^3 x}dx$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$$I = \int\frac{dt}{3t-4t^3}$$

$$= \int\frac{1}{t^3\left(\frac{3}{t^2}-4\right)}dt$$

Let $\frac{3}{t^2} - 4 = z \Rightarrow -\frac{6}{t^3}dt = dz$

$$I = -\frac{1}{6}\int\frac{dz}{z}$$

$$= -\frac{1}{6}\log|z| + C$$

$$= -\frac{1}{6}\log|3\,\text{cosec}^2 x - 4| + C$$

30. Let $I = \int x^2 \log(x^2+1)dx$

$$= \log(x^2+1)\cdot\frac{x^3}{3} - \int \frac{2x}{x^2+1}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx$$

$$= \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$$

$$= \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\left[\frac{x^3}{3} - x + \tan^{-1}x\right] + C$$

31. Let $I = \int_1^4 \frac{1}{\sqrt{2x+1} - \sqrt{2x-1}}dx$

$$= \int_1^4 \frac{\sqrt{2x+1} + \sqrt{(2x-1)}}{2}\,dx$$

$$= \left.\frac{(2x+1)^{3/2}}{3\times 2} + \frac{(2x-1)^{3/2}}{3\times 2}\right|_1^4$$

$$= \left(\frac{27}{6} + \frac{7^{3/2}}{6}\right) - \left(\frac{3^{3/2}}{6} + \frac{1}{6}\right)$$

$$= \frac{26 + 7^{3/2} - 3^{3/2}}{6} \text{ or } \frac{26 + 7\sqrt{7} - 3\sqrt{3}}{6}$$

32. Let $e^x = t$. Then $e^x\,dx = dt$

Given integral becomes

$$\int \frac{dt}{\sqrt{t^2 - 4t - 5}} = \int \frac{dt}{\sqrt{(t-2)^2 - 3^2}}$$

$$= \log|(t-2) + \sqrt{t^2 - 4t - 5}| + C$$

$$= \log|e^x - 2 + \sqrt{e^{2x} - 4e^x - 5}| + C$$

33. $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^{100}x}{\sin^{100}x + \cos^{100}x}dx$

$$I = 2\int_0^{\pi/2} \frac{\sin^{100}x}{\sin^{100}x + \cos^{100}x} \quad \text{...(i)}$$

as $f(x) = \frac{\sin^{100}x}{\sin^{100}x + \cos^{100}x}$ is even

$$I = 2\int_0^{\pi/2} \frac{\cos^{100}x}{\cos^{100}x + \sin^{100}x}\,dx \quad \text{...(ii)}$$

using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

On adding eqs. (i) & (ii),

$$2I = 2\int_0^{\pi/2} \frac{\sin^{100}x + \cos^{100}x}{\cos^{100}x + \sin^{100}x} = 2\int_0^{\pi/2} dx$$

$$I = x|_0^{\pi/2} \Rightarrow I = \frac{\pi}{2}$$

34. Let $I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{x} + \sqrt{4-x}}dx$ ...(i)

$$I = \int_1^3 \frac{\sqrt{4-(4-x)}}{\sqrt{x} + \sqrt{4-x}}dx$$

$$= \int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}}\,dx \quad \text{...(ii)}$$

$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$ On adding eqs. (i) & (ii), we get

$$2I = \int_1^3 \frac{\sqrt{4-x} + \sqrt{x}}{\sqrt{4-x} + \sqrt{x}} = \int_1^3 1dx$$

$$2I = x|_1^3 = 2$$

$$\Rightarrow \quad I = 1$$

35. Let $I = \int_1^e \frac{1}{\sqrt{4x^2 - (x\log x)^2}}\,dx$

$$= \int_1^e \frac{1}{x\sqrt{4 - (\log x)^2}}\,dx$$

$\left[\text{Let } \log x = t \Rightarrow \frac{1}{x}dx = dt\right]$

$$= \int_0^1 \frac{dt}{\sqrt{4 - t^2}}$$

$$= \sin^{-1}\left.\frac{t}{2}\right|_0^1 = \frac{\pi}{6}$$

36. Let $I = \int \frac{x^2 + x + 1}{(x+1)^2(x+2)}\,dx$

Here $\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$

$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

On comparing, we get

$A = -2$, $B = 1$ and $C = 3$

$$\therefore \quad I = \int \frac{-2dx}{x+1} + \int \frac{dx}{(x+1)^2} + 3\int \frac{dx}{x+2}$$

$$= -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + C$$

37. $I = \int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1 - \sin 2x}{1 - \cos 2x}\right)dx$

Put $2x = t$ so that $2dx = dt$

When $x = \frac{\pi}{2}$, $t = \pi$; $x = \frac{\pi}{4}$, $t = \frac{\pi}{2}$

Thus, $I = \int_{\pi/2}^{\pi} e^t\left(\frac{1 - \sin t}{1 - \cos t}\right)\frac{dt}{2}$

$$= \int_{\pi/2}^{\pi} e^t \left(\frac{1 - 2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} \right) \frac{dt}{2}$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} e^t \left(\frac{1}{2}\ \text{cosec}^2\frac{t}{2} - \cot\frac{t}{2} \right) dt$$

$$= -\frac{1}{2}\left| e^t \cot\frac{t}{2} \right|_{\pi/2}^{\pi}$$

$$\left[\because \int e^x f(x) + f'(x)\, dx = e^x f(x) + c \right]$$

$$= \frac{1}{2} e^{\pi/2}$$

38. $$I = \int_{-2}^{2} \frac{x^2}{1+5^x} dx \qquad ...(1)$$

Applying property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-2}^{2} \frac{(-2+2-x)^2}{1+5^{(-2+2-x)}} dx$$

$$I = \int_{-2}^{2} \frac{x^2 5^x}{1+5^x} dx \qquad ...(2)$$

Adding (1) and (2)

$$2I = \int_{-2}^{2} \frac{x^2(1+5^x)}{1+5^x} dx$$

$$= \int_{-2}^{2} x^2\, dx$$

$$= \left| \frac{x^3}{3} \right|_{-2}^{2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

$$\Rightarrow \qquad \text{I} = \frac{16}{6} \text{ or } \frac{8}{3}$$

39. $$I = \int_0^{\pi/2} \sqrt{\sin x}.(1 - \sin^2 x)^2 \cos x\, dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$= \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$= \int_0^1 (\sqrt{t} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}})\, dt$$

$$= \left[\frac{2t^{3/2}}{3} + \frac{2t^{11/2}}{11} - \frac{4t^{7/2}}{7} \right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{64}{231}$$

40. Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$\Rightarrow A = 1, B = 1, C = 1$

Hence, $I = \int \frac{2}{(1-x)(1-x^2)} dx$

$$= \int \left[\frac{1}{1-x} + \frac{x+1}{1+x^2} \right] dx$$

$$= \int \frac{1}{1-x} dx + \int \frac{x+1}{x^2+1} dx$$

$$= \int \frac{1}{1-x} dx + \frac{1}{2}\int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$$

$$= -\log|1-x| + \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$$

41. $$I = \int_{1/3}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx$$

$$= \int_{1/3}^{1} \frac{(x^3)^{1/3}\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4} dx$$

$$= \int_{1/3}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3} dx$$

Put $\left(\frac{1}{x^2} - 1\right) = t$ so that $\frac{-2}{x^3} dx = dt$

Thus, $$I = \int_8^0 t^{\frac{1}{3}} \times \frac{dt}{(-2)}$$

$$I = \frac{1}{2}\int_0^8 t^{\frac{1}{3}} dt$$

$$= \left[\frac{1}{2} \cdot \frac{3}{4} t^{4/3} \right]_0^8$$

$$= \frac{48}{8} = 6$$

42. $\int_1^3 (|x-1| + |x-2|)\, dx$

$$= \int_1^3 |x-1|\, dx + \int_1^3 |x-2|\, dx$$

$$= \int_1^3 (x-1)\, dx - \int_1^2 (x-2)\, dx + \int_2^3 (x-2)\, dx$$

$$= \left| \frac{(x-1)^2}{2} \right|_1^3 - \left| \frac{(x-2)^2}{2} \right|_1^2 + \left| \frac{(x-2)^2}{2} \right|_2^3$$

$$= (2) - \left(0 - \frac{1}{2}\right) + \frac{1}{2} = 3$$

43. Let $$I = \frac{x^2}{x^2+6x+12} dx$$

$$= \int \left| 1 - \frac{6x+12}{x^2+6x+12} \right| dx$$

$$= \int 1\, dx - \int \frac{6x+12}{x^2+6x+12} dx$$

$$= x - 6\int \frac{x+2}{x^2+6x+12} dx \qquad ...(1)$$

Let $\qquad I_1 = \int \frac{x+2}{x^2+6x+12}\,dx$

$$x + 2 = A\frac{d}{dx}(x^2+6x+12)+B$$

$$\Rightarrow \quad x + 2 = A(2x + 6) + B$$

$\Rightarrow A = \frac{1}{2}$ and $B = -1$

So, $\qquad I_1 = \frac{1}{2}\int \frac{2x+6}{x^2+6x+12}\,dx - \int \frac{1}{x^2+6x+12}\,dx$

$$= \frac{1}{2}\log|x^2+6x+12| - \int \frac{1}{(x+3)^2+(\sqrt{3})^2}\,dx$$

$$= \frac{1}{2}\log|x^2+6x+12| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)+C$$

From (1), we have

$$I = x - 3\log|x^2+6x+12| + 2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)+C$$

44. Let $\qquad I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx \qquad$ ...(i)

$$= \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - x)}}\,dx$$

Apply property:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}\,dx$$

$$= \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx \qquad \text{...(ii)}$$

On adding eqs. (i) and (ii),

$$\Rightarrow \quad 2I = \int_0^{2\pi} \frac{e^{\sin x}+1}{e^{\sin x}+1}\,dx$$

$$= \int_0^{2\pi} 1.dx = 2\pi$$

$\Rightarrow I = \pi$

45. $I = \int \frac{x^4}{(x-1)(x^2+1)}\,dx$

$$= \int\left[x+1+\frac{1}{(x-1)(x^2+1)}\right]dx$$

$$= \int x\,dx + \int 1\,dx + \int \frac{1}{(x-1)(x^2+1)}\,dx$$

$$= \frac{x^2}{2}+x+\int\left[\frac{1}{2(x-1)} - \frac{1}{2}\frac{(x+1)}{(x^2+1)}\right]dx$$

$$= \frac{x^2}{2}+x+\frac{1}{2}\int\frac{1}{(x-1)}\,dx - \frac{1}{2}\int\frac{x}{x^2+1}\,dx - \frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

(Using partial fractions)

$$= \frac{x^2}{2}+x+\frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

46. $\qquad I = \int_{-\pi/4}^{\pi/4} \frac{\cos 2x}{1+\cos 2x}\,dx$

$$= 2\int_0^{\pi/4} \frac{\cos 2x}{1+\cos 2x}\,dx$$

$$= 2\int_0^{\pi/4}\left(1-\frac{1}{1+\cos 2x}\right)dx$$

$$= 2\int_0^{\pi/4}\left(1-\frac{1}{2\cos^2 x}\right)dx$$

$$= 2\int_0^{\pi/4}\left(1-\frac{1}{2}\sec^2 x\right)dx$$

$$= (2x - \tan x)\Big|_0^{\pi/4}$$

$$= \left(\frac{\pi}{2}-1\right)$$

47. Let $\qquad I = \int e^{x^2}(x^5+2x^3)\,dx$

$$= \int e^{x^2}.x(x^4+2x^2)dx$$

Put $x^2 = t$ so that $2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int e^t(t^2+2t)dt$$

$$= \frac{1}{2}e^t.t^2$$

$$\left[\because \int e^x[f(x)+f'(x)]\,dx = e^x f(x)+C\right]$$

$$= \frac{1}{2}e^{x^2}(x^4)+C$$

48. $\int e^x \frac{x-5}{(x-2)^4}\,dx = \int e^x\left[\frac{(x-2)-3}{(x-2)^4}\right]dx$

$$= \int e^x\left[\frac{1}{(x-2)^3} - \frac{3}{(x-2)^4}\right]dx$$

Applying property,

$$\int e^x[f(x)+\int f'(x)]\,dx = e^x\int f(x)+C$$

$$= \frac{e^x}{(x-2)^3}+C$$

49. $\int \frac{2}{(1-x)(1+x^2)}\,dx = \int\frac{1}{1-x}\,dx + \int\frac{x+1}{1+x^2}\,dx$

$$= \int\frac{1}{1-x}\,dx + \frac{1}{2}\int\frac{2x}{1+x^2}\,dx + \int\frac{1}{1+x^2}\,dx$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C$$

50. $\int \sqrt{3-2x-x^2}\;dx$

$$= \int \sqrt{2^2-(x+1)^2}\;dx$$

$$= \frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\frac{x+1}{2} + C$$

[Using formula, $\int\sqrt{a^2-x^2}\,dx$

$$= \frac{x}{2}\sqrt{a^2-x^2} \quad + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C]$$

51. Let $\quad I = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$

$$I = \int_0^{\pi}\frac{x\sin x}{1+\sin x}dx, \quad \text{...(i)}$$

$$I = \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x} \quad \text{...(ii)}$$

[$\because \sin(\pi - x) = \sin x$]

$$\Rightarrow \quad 2I = \pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx \text{ [On adding eqs. (i) \& (ii)]}$$

$$= \pi\int_0^{\pi}\left[1-\frac{1}{1+\sin x}\right]dx$$

$$= \pi\int_0^{\pi}\left[1-\frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi}\left[1-\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]dx$$

$$= \pi\int_0^{\pi}\left[1-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]dx$$

$$\Rightarrow \quad 2I = \pi\left[x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^{\pi}$$

$$= \pi(\pi-2)$$

$$\therefore I = \frac{\pi}{2}(\pi-2)$$

52. $\quad I = \int\frac{\cos\theta}{\sqrt{3-3\sin\theta-\cos^2\theta}}d\theta$

$$= \int\frac{\cos\theta}{\sqrt{\sin^2\theta-3\sin\theta+2}}d\theta$$

[$\because \cos^2\theta = 1 - \sin^2\theta$]

Putting $\sin\theta = t$ gives

$$I = \int\frac{dt}{\sqrt{t^2-3t+2}}$$

$$= \int\frac{dt}{\sqrt{\left(t-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \log\left|\left(t-\frac{3}{2}\right)+\sqrt{t^2-3t+2}\right|+C$$

$$= \log\left|\left(\sin\theta-\frac{3}{2}\right)+\sqrt{\sin^2\theta-3\sin\theta+2}\right|+C$$

53. $\quad I = \int_0^{\pi/2}\frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx \quad \text{...(1)}$

using $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}$$

$$\Rightarrow \quad I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x+\sin^4 x}dx \quad \text{...(2)}$$

Adding (1) and (2)

$$2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$= \pi\int_0^{\pi/4}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/4}\frac{\tan x\sec^2 x}{(\tan^2 x)^2+1}dx$$

($\because$ dividing by $\cos^4 x$)

Putting $\tan^2 x = t$ gives $2\tan x\sec^2 x\,dx = dt$

$$I = \frac{\pi}{4}\int_0^1\frac{1}{t^2+1}dt$$

$$\Rightarrow I = \frac{\pi}{4}[\tan^{-1}t]_0^1 = \frac{\pi^2}{16}$$

54. $\quad I = \int_1^3(|x-1|+|x-2|)dx$

$$= \int_1^2[(x-1)-(x-2)]dx + \int_2^3[(x-1)+(x-2)]dx$$

$$= \int_1^2 1\,dx + \int_2^3(2x-3)\,dx$$

$$= [x]_1^2 + [x^2-3x]_2^3$$

$$= 1 + 2 = 3$$

55. $\quad I = \int\frac{x}{(x^2+1)(x-1)}dx$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1}+\frac{C}{x-1}$

$\Rightarrow x = (x-1)(Ax+B) + C(x^2+1)$

This gives $A = -\frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{2}$

$$\therefore \; I = -\frac{1}{2}\int\frac{x-1}{x^2+1}dx + \frac{1}{2}\int\frac{1}{x-1}dx$$

$$= -\frac{1}{2}\left\{\frac{1}{2}\int\frac{2x}{x^2+1}dx - \int\frac{1}{x^2+1}dx\right\} + \frac{1}{2}\int\frac{1}{x-1}dx$$

$$= -\frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

56. $$I = \int \frac{2x^2+1}{x^2(x^2+4)}\,dx$$

Let $\dfrac{2x^2+1}{x^2(x^2+4)} = \dfrac{2y+1}{y(y+4)}$, where $x^2 = y$

Put $\dfrac{2y+1}{y(y+4)} = \dfrac{A}{y} + \dfrac{B}{y+4}$

$\Rightarrow$ $2y + 1 = A(y + 4) + By$

$\Rightarrow A = \dfrac{1}{4}, B = \dfrac{7}{4}$

$\therefore \dfrac{2y+1}{y(y+4)} = \dfrac{1}{4y} + \dfrac{7}{4(y+4)} = \dfrac{1}{4x^2} + \dfrac{7}{4(x^2+4)}$

$\Rightarrow$ $$I = \frac{1}{4}\int \frac{1}{x^2}dx + \frac{7}{4}\int \frac{1}{x^2+4}dx$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

57. $$I = \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

Let $\sqrt{x} = y^3 \Rightarrow x = y^6$ so that $dx = 6y^5dy$

$$= \int \frac{6y^5dy}{y^3+y^2}$$

$$= 6\int \frac{y^3}{y+1}dy$$

$$= 6\int \left[(y^2 - y + 1) - \frac{1}{y+1}\right]dy$$

$$= 6\left[\frac{y^3}{3} - \frac{y^2}{2} + y - \log|y+1|\right] + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6x^{1/6} - 6\log(x^{1/6}+1) + C$$

58. Let $\sin x = t$, then $\cos x\, dx = dt$

$$\therefore \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(4+\sin x)}dx$$

$$= \int_0^1 \frac{dt}{(1+t)(4+t)}$$

$$= \frac{1}{3}\left[\int_0^1 \frac{1}{1+t}dt - \int_0^1 \frac{1}{4+t}dt\right]$$

$$= \frac{1}{3}\left[\log(1+t)\big|_0^1 - \log(4+t)\big|_0^1\right]$$

$$= \frac{1}{3}[\log 2 - \log 5 + \log 4] \qquad [\because \log 1 = 0]$$

or $$= \frac{1}{3}\log\frac{8}{5}$$

59. $$I = \int \frac{x^3+x}{x^4-9}dx$$

$$= \int \frac{x^3}{x^4-9}dx + \int \frac{x}{x^4-9}dx$$

$$= \frac{1}{4}\int \frac{4x^3}{x^4-9}dx + \frac{1}{2}\int \frac{dt}{t^2-3^2}, \text{ where } x^2 = t$$

$$= \frac{1}{4}\log|x^4-9| + \frac{1}{2}\left[\frac{1}{2(3)}\log\left|\frac{t-3}{t+3}\right|\right] + C$$

$$= \frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C$$

60. Consider $\displaystyle\int(\tan^{-1}x).dx = \tan^{-1}x.x - \int \frac{1}{1+x^2}\cdot x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

Now, $\displaystyle\int_0^1 (\tan^{-1}x)\,dx = \left[x\tan^{-1}x - \frac{1}{2}\log|1+x^2|\right]_0^1$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

61. $$I = \int \frac{2x.dx}{x^2+3x+2}$$

$$= \int \frac{2x}{(x+1)(x+2)}dx$$

$$= \int \frac{-2}{x+1}dx + \int \frac{4\,dx}{x+2} \quad \text{(using partial fraction)}$$

$= -2\log|x+1| + 4\log|x+2| + C$

62. Here, $x^3 - x \geq 0$ on $[-1, 0]$, $x^3 - x \leq 0$ on $[0, 1]$ and $x^3 - x \geq 0$ on $(1, 2)$

+   −   +

−1   0   1   2

So, $$I = \int_{-1}^0 (x^3-x)dx + \int_0^1 (x-x^3)dx + \int_1^2 (x^3-x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{9}{4}$$

$$= \frac{11}{4}$$

63. $$I = \int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)dx$$

$f(x) = \sin|x| + \cos|x|$

$f(x)$ is an even function

$$I = 2\int_0^{\pi/2} (\sin|x| + \cos|x|)dx$$

$$= 2\int_0^{\pi/2} (\sin x + \cos x)\,dx$$

$$= 2[[-\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}]$$

$= 2[1 + 1]$

$= 4$

64. $I = \int_1^4 \{|x|+|3-x|\}\,dx$

$$= \int_1^3 \{|x|+|3-x|\}dx + \int_3^4 \{|x|+|3-x|\}\,dx$$

$$= \int_1^3 3\,dx + \int_3^4 (2x-3)\,dx$$

$$= |3x|_1^3 + |x^2-3x|_3^4$$

$$= 6 + 4 = 10$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x}\,dx$

Put $\sin x - \cos x = t$, so that $(\cos x + \sin x)\,dx = dt$

Now, $(\sin x - \cos x)^2 = t^2$

$\sin^2 x + \cos^2 x - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$

$$I = \int_{-1}^0 \frac{dt}{25-16t^2}$$

$$= \frac{1}{16}\int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{40}\left[\log\left(\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right)\right]_{-1}^0$$

$$= \frac{1}{40}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^0$$

$$= \frac{1}{40}\left[\log 1 - \log\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{40}\log 9 \text{ or } \frac{1}{20}\log 3$$

2. Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx$

$$= 2\int_0^{\pi/2} \sin x \cos x \tan^{-1}(\sin x)dx$$

Put $\sin x = t$ so that $\cos x\,dx = dt$

$$I = 2\int_0^1 t\tan^{-1} t\,dt$$

$$= 2\left[\tan^{-1} t\left(\frac{t^2}{2}\right)\right]_0^1 - \frac{1}{2}\int_0^1\left[\frac{t^2}{1+t^2}dt\right]$$

$$= 2\left[\left\{\tan^{-1} t\left(\frac{t^2}{2}\right)\right\}_0^1 - \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \frac{1}{1+t^2}\,dt\right]$$

$$= 2\left|\left(\frac{t^2}{2}\right)\tan^{-1} t - \frac{1}{2}t + \frac{1}{2}\tan^{-1} t\right|_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$$

3. $\int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int_0^{\pi/2} e^x\left(\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$

$$= \int_0^{\pi/2} e^x\left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right)dx$$

On applying $\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$

$$= \left[e^x \tan\frac{x}{2}\right]_0^{\pi/2}$$

$$= e^{\frac{\pi}{2}}$$

4. $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

Put $\sin x - \cos x = t$ so that $(\cos x + \sin x)\,dx = dt$

$$= \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

[$\because$ $(\sin x - \cos x)^2 = t^2$
$\sin^2 x + \cos^2 x - \sin 2x = t^2$
$1 - \sin 2x = t^2$
$\sin 2x = 1 - t^2$]

$$= [\sin^{-1} t]_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}}$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

5. $\int \frac{(3\cos x - 2)\sin x}{5 - \sin^2 x - 4\cos x}\,dx$

$$= \int \frac{(3\cos x - 2)\sin x}{5 - (1-\cos^2 x) - 4\cos x}\,dx$$

Put $\cos x = t$ so that, $-\sin x\,dx = dt$

$$= \int \frac{2-3t}{5-(1-t^2)-4t}\,dt$$

$$= \int \frac{2-3t}{(t-2)^2}\,dt$$

Now, $\int \frac{2-3t}{(t-2)^2}\,dt = -3\int \frac{1}{t-2}dt - 4\int \frac{1}{(t-2)^2}\,dt$

[By partial fraction]

$$= -3\log|t-2| - 4\left(\frac{-1}{t-2}\right) + C$$

$$= -3\log|\cos x - 2| + \frac{4}{\cos x - 2} + C$$

6. $$I = \int_{-2}^{2} \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4} dx$$

$$= \int_{-2}^{2} \frac{x^3}{x^2 + 4|x| + 1} dx + \int_{-2}^{2} \frac{|x| + 1}{x^2 + 4|x| + 4} dx$$

$$= I_1 + I_2 (\text{say})$$

$$I_1 = 0 \quad \left(\because \frac{x^3}{x^2 + 4|x| + 4} \text{ is an odd function}\right)$$

$$\therefore \quad I = 2\int_0^2 \frac{x+1}{(x+2)^2} dx$$

Put $x + 2 = t$, so that $dx = dt$

$$= 2\int_2^4 \frac{t-1}{t^2} dt$$

$$= 2\left[\int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) dt\right]$$

$$= 2\left[\log|t| + \frac{1}{t}\right]_2^4$$

$$= 2\left[\log 4 + \frac{1}{4} - \log 2 - \frac{1}{2}\right]$$

$$= 2\log 2 - \frac{1}{2}$$

7. $$I = \int_0^{\pi} \frac{x}{1+\sin x} dx \quad \text{...(i)}$$

$$I = \int_0^{\pi} \frac{\pi - x}{1 + \sin(\pi - x)} dx$$

$$I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx$$

$$I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x} dx \quad \text{...(ii)}$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{dx}{1+\sin x}$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{dx}{1+\sin x}$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{\pi}{2} \int_0^{\pi} \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{\pi}{4} \left| -2\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right|_0^{\pi}$$

$$= -\frac{\pi}{2}\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right]$$

$$= \pi$$

8. $$\int_0^{\pi/2} (2\log\cos x - \log\sin 2x) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\cos^2 x}{2\sin x \cos x}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\cot x}{2}\right) dx = I \text{ (say)} \quad \text{...(i)}$$

$$I = \int_0^{\pi/2} \log\frac{\tan x}{2} dx \quad \text{...(ii)}$$

(using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$)

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \log\left(\frac{\tan x}{2} \frac{\cot x}{2}\right) dx$$

$$2I = \log\left(\frac{1}{4}\right)[x]_0^{\pi/2}$$

$$I = \frac{\pi}{4}\log\frac{1}{4} \text{ or } -\frac{\pi}{2}\log 2$$

9. Let, $$I = \int_0^{\pi} \frac{x}{9\sin^2 x + 16\cos^2 x} dx \quad \text{...(i)}$$

$$I = \int_0^{\pi} \frac{\pi - x}{9\sin^2 x + 16\cos^2 x} dx$$

Also, $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$

(using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$) ...(ii)

Adding equation (i) and (ii)

$$2I = \int_0^{\pi} \frac{\pi}{9\sin^2 x + 16\cos^2 x} dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{1}{9\sin^2 x + 16\cos^2 x} dx$$

$$I = \pi\left[\int_0^{\pi/4} \frac{\sec^2 x}{9\tan^2 x + 16} dx + \int_{\pi/4}^{\pi/2} \frac{\text{cosec}^2 x}{9 + 16\cot^2 x} dx\right]$$

Using properly $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$= \pi[I_1 + I_2] \text{ (say)}$$

$$= \pi\left[\int_0^1 \frac{dt}{9t^2 + 16} - \int_1^0 \frac{dz}{9 + 16z^2}\right]$$

($t = \tan x$ in I_1, $z = \cot x$ in I_2)

$$= \frac{\pi}{12}\left\{\left[\tan^{-1}\frac{3t}{4}\right]_0^1 - \left[\tan^{-1}\frac{4z}{3}\right]_1^0\right\}$$

$$= \frac{\pi}{12}\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{4}{3}\right) \quad \left(\because \tan^{-1}\frac{4}{3} = \cot^{-1}\frac{3}{4}\right)$$

$$= \frac{\pi}{12} \times \frac{\pi}{2} = \frac{\pi^2}{24} \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

10. Let,
$$I = \int_{-3}^{3} \frac{x^4}{1+e^x}\,dx \quad \text{...(i)}$$

$$I = \int_{-3}^{3} \frac{x^4}{1+e^{-x}}\,dx$$

(using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$)

$$= \int_{-3}^{3} \frac{e^x x^4}{1+e^x}\,dx \quad \text{...(ii)}$$

Adding equations (i) and (ii)

$$2I = \int_{-3}^{3} \frac{x^4(1+e^x)}{1+e^x}\,dx$$

$$\Rightarrow \quad 2I = \left.\frac{x^5}{5}\right|_{-3}^{3}$$

$$\Rightarrow \quad I = \frac{243}{5}$$

Level - 2 ADVANCED COMPETENCY FOCUSED QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQs) (1 Marks)

1. **Option (D) is correct.**

Explanation: We are given:

$$P = \int_a^b f(x)dx$$

Statement 1: If $f(x)$ is well-defined and continuous in (a, b), then P is always positive.

This is false because the definite integral computes the net area under the curve. If $f(x)$ takes both positive and negative values, the integral might be zero or negative. For example, if $f(x) = x$ over $(-1, 1)$, the integral evaluates to zero.

Statement 2: If P exists, then $f(x)$ is always continuous in (a, b).

This is false because a function can be integrable but not continuous. Discontinuous functions like piecewise or step functions (e.g., the Dirichlet function restricted to a finite interval) can have well defined integrals.

For example, $f(x) = 1$ for $x \neq 0$, and $f(0) = 0$ is integrable over $(-1, 1)$ but discontinuous.

Both Statement 1 and Statement 2 are false.

2. **Option (C) is correct.**

Explanation: The given integral is:

$$I = \int_{-3}^{3} \sqrt{x^2 - 4}\,dx$$

We analyze whether the integral is properly defined:

Domain of $\sqrt{x^2-4}$:

The function inside the square root must be non-negative for real values.

This requires $x^2 - 4 \geq 0$, which simplifies to:

$$x^2 \geq 4 \Rightarrow x \leq -2 \text{ or } x \geq 2$$

This means that $f(x) = \sqrt{x^2-4}$ is not defined for $x \in (-2, 2)$, making the integral improper.

Since the integral is undefined over $(-2, 2)$, Varath's statement is incorrect.

The fundamental theorem of calculus requires the function to be continuous over the interval for direct evaluation, which fails here.

Thus, the integral is not defined over $[-3, 3]$.

It is false, as the integral is not defined over the interval.

3. **Option (A) is correct.**

Explanation: Total water $= \int_0^5 4t\,dt$

$$= 4\int_0^5 t\,dt = 4\left[\frac{t^2}{2}\right]_0^5$$

$$= 4 \cdot \frac{25}{2} = 50 \text{ litres}$$

4. **Option (A) is correct.**

Explanation: $C(x) = \int (2x+3)dx = 2 \cdot \frac{x^2}{2} + 3x$

$$= x^2 + 3x$$

So, the variable cost for 10 units:

$$C(10) = 10^2 + 3 \times 10 = 100 + 30 = ₹\,130$$

Total cost = Variable cost + Fixed cost

= 130 + 500 = ₹ 630

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (C) is correct.**

Explanation: Assertion is true. The total distance travelled by a vehicle can indeed be found by integrating the velocity function over time (specifically, the speed, which is the absolute value of velocity, if direction changes are involved).

$$\text{Distance} = \int_{t_1}^{t_2} |v(t)|\,dt$$

Reason is false because this is a definition of differentiation, not integration. Integration does not give the rate of change — it gives the net accumulation of a quantity over an interval.

2. **Option (A) is correct.**

Explanation: Assertion is true. The total work done by a machine, when the force varies with displacement, is calculated using a definite integral:

$$W = \int_a^b F(x)\, dx$$

where F(x) is the variable force applied over displacement from $x = a$ to $x = b$

Reason is also true. The definite integral of a force function over displacement does indeed give the total work done.

3. **Option (A) is correct.**

Explanation: Assertion is true. Integration is used to calculate volumes of solids, including water tanks with curved surfaces such as cylinders, spheres, and cones, by revolving curves or integrating cross-sectional areas.

Reason is also true because definite integrals are a powerful mathematical tool to calculate the volumes of irregular solids by integrating over the required bounds.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. Ankit is incorrect.

Ankit made an error in step 5.

For example, on substituting cos x as t, one also has to substitute dx in terms of dt, which Ankit has not done. Instead, he has simply replaced dx with dt.

2. Compares the integral with the standard form of integration by parts to conclude:

$$\int g(x)dx = \sin x$$

Uses the above step to find $g(x)$ as cos x or cos $(2n\pi \pm x)$ where n is a whole number.

3. Rewrites the integral as:

$$2\int \frac{3x^2+2}{x^3+2x} dx$$

Substitutes $x^3 + 2x$ as t to get:

$$dx = \frac{dt}{3x^2+2}$$

Rewrites the integral as:

$$2\int \frac{1}{t}\, dt$$

Integrates the above expression and gets 2 log $|t|$ + C as the solution where C is an arbitrary constant.

Substitutes t as $x^3 + 2x$ to get 2 log $|x^3 + 2x|$ + C as the solution.

4. Writes that Lalitha made an error in step 4.

Writes that Lalitha has integrated the expression given in step 3 as cosec θ instead of –cosec θ.

5. On differentiating log |log x| + C using the chain rule, we get $f(x) = \frac{1}{x \log x}$.

6. The value of the definite integral as 0.

Since, we are integrating the function from $x = 3$ to $x = 3$, the area under the function will be zero.

7. By differentiating the RHS of the given equation using the differentation of $\tan^{-1}x$ as follows:

$$\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}\left(\frac{x-4}{3}\right)\right) = \frac{1}{9+(x-4)^2}$$

$$f(x) = 9 + (x-4)^2$$

8. Rewrites the given expression as:

$$\frac{e^6}{x}$$

Integrates the above expression with respect to x as:

$$e^6 \ln |x| + C$$

where, C is the constant of integration.

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. Differentiates both sides of the given equation with respect to x to get:

$$-9\, x\, (e^{-3x}) = (e^{-3x})(3p\, x^2 + q) + (px^3 + qx + r)(-3e^{-3x})$$

Simplifies the above equation as:

$$-9x\, (e^{-3x}) = (e^{-3x})(-3p\, x^3 + 3p\, x^2 - 3q\, x + q - 3r)$$

Equates the coefficients on both sides of the equation to find the values of p, q and r as 0, 3 and 1 respectively.

Writes that the constant, s cannot be uniquely determined from the given information.

(Award full marks if the problem is solved by applying integration by parts to the LHS and then equating the coefficients on both sides.)

2. Substitute $\left(\frac{1}{m} - 1\right) = u$, we get:

$$du = -\frac{1}{m^2}\, dm$$

Rewrites the given integral as:

$$-\int \cos^2 u\, du$$

Substitutes:

$$\cos^2 u = \frac{(1+\cos 2u)}{2}$$

In the above integral and rewrites it as follows:

$$I = -\frac{1}{2}\int (1+\cos 2u)\, du$$

Integrates the above integral to get the following expression where C is the arbitrary constant:

$$I = -\frac{1}{2}\left(u+\frac{1}{2}\sin 2u\right)+C$$

Substitute u as $\left(\frac{1}{m}-1\right)$ in the above expression to get the following expression as the solution:

$$I = -\frac{1}{2}\left[\frac{1}{m}-1+\frac{1}{2}\sin 2\left(\frac{1}{m}-1\right)\right]+C$$

3. $$\int_0^{30}(200-2x)\,dx = \left[200x-x^2\right]_0^{30}$$

Substituting the limits:

$$= (200\times 30-30^2)-(200\times 0-0^2)$$
$$= (6000-900)-0 = 5100$$

5100 items were sold in the first 30 days.

4. $$\text{Distance} = \int_0^4 v(t)\,dt$$
$$= \int_0^4 (5t^2-2t+3)\,dt$$
$$= \left[\frac{5t^3}{3}-t^2+3t\right]_0^4$$

At $t = 4$:

$$= \frac{5(4)^3}{3}-(4)^2+3(4)$$
$$= \frac{320}{3}-16+12$$
$$= \frac{320}{3}-4 = \frac{320-12}{3} = \frac{308}{3}$$

$$\text{Total distance} = \frac{308}{3} = 102.67\text{ m}$$

5. $$V = \int_0^6 R(t)\,dt = \int_0^6 (6t-0.5t^2)\,dt$$
$$= \left[3t^2-\frac{0.5t^3}{3}\right]_0^6 = \left[3t^2-\frac{t^3}{6}\right]_0^6$$

At $t = 6$:

$$V = 3(6)^2-\frac{(6)^3}{6} = 3(36)-\frac{216}{6} = 108-36 = 72$$

Total volume = 72 litres

CASE BASED QUESTIONS (4 Marks)

1. (i) (a) The velocity function of the car (in m/s) is given as:

$$v(t) = \int a(t)dt \text{ or } -\frac{1}{3}\int t\,dt$$

Integrates the above expression to get $v(t)$ as follows:

$$v(t) = -\frac{1}{3}\times\frac{t^2}{2}+C$$

where C is an arbitrary constant.

At $t = 0$, $v(0) = 15$ m/s.

$$15 = -\frac{1}{3}\times\frac{0}{2}+c \Rightarrow c = 15$$

$$\therefore \quad v(t) = -\frac{t^2}{6}+15$$

(b) The velocity of the car will be zero when the car stops. Hence, get the equation:

$$0 = 15-\frac{t^2}{6}$$

$$t = \sqrt{90} \text{ or } 3\sqrt{10} \text{ seconds.}$$

(ii) The displacement function of the car (in m) is given as:

$$x(t) = \int v(t)\,dt \text{ or } \int\left(15-\frac{t^2}{6}\right)dt$$

Integrates the above expression to get:

$$x(t) = 15t-\frac{t^3}{18}+C$$

where C is an arbitrary constant.

At $t = 0$, the car starts decelerating until it stops and $x(0) = 0$.

Substitutes these values in the above expression to get C as 0 and rewrites the displacement function $x(t)$ as:

$$x(t) = 15t-\frac{t^3}{18}$$

The displacement from the moment it starts declerating until it stops is given as:

$$x(3\sqrt{10}) = 15\times 3\sqrt{10}-\frac{(3\sqrt{10})^3}{18}$$
$$= 45\sqrt{10}-15\sqrt{10}$$
$$= 30\sqrt{10}\text{ m}$$

2. (i) It represents the rate of electricity generation in kilowatts at time t hours after sunrise.

(ii) The maximum value of $\sin\left(\frac{\pi t}{12}\right)$ is 1, which occurs at $t = 6$ (i.e., noon).

Maximum power = 100 × 1 = 100 = 100 kW.

(iii) $$\text{Total energy} = \int_0^{12} 100\sin\left(\frac{\pi t}{12}\right)dt$$
$$= 100\int_0^{12}\sin\left(\frac{\pi t}{12}\right)dt$$

Let $u = \frac{\pi t}{12} \Rightarrow dt = \frac{12}{\pi} du$

When $t = 0, u = 0$

When $t = 12, u = \pi$

Total energy $= 100 \cdot \frac{12}{\pi} \int_0^{\pi} \sin u \, du$

$= \frac{1200}{\pi} [-\cos u]_0^{\pi}$

$= \frac{1200}{\pi} [-\cos(\pi) + \cos(0)]$

$= \frac{1200}{\pi} [-(-1) + 1]$

$= \frac{1200}{\pi}(2) = \frac{2400}{\pi}$

≈ 763.64 kWh

OR

Avg Power $= \frac{1}{12} \int_0^{12} E(t) dt = \frac{1}{12} \cdot \frac{2400}{\pi}$

$= \frac{200}{\pi} \approx 63.64$ kW

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. Uses integration by parts to rewrite the given integral as:

$$\cot^{-1}\left(\frac{5}{x}\right)\int dx - \left[\frac{d}{dx}\left(\cot^{-1}\left(\frac{5}{x}\right)\right)\int dx\right] dx$$

Simplifies the differentiation of $\cot^{-1}\left(\frac{5}{x}\right)$ in the above expression as:

$$\frac{d}{dx}\cot^{-1}\left(\frac{5}{x}\right) = -\frac{1}{1+\left(\frac{5}{x}\right)^2}\frac{d}{dx}\left(\frac{5}{x}\right)$$

$$= \frac{\frac{5}{x^2}}{1+\left(\frac{5}{x}\right)^2}$$

$$= \frac{5}{x^2+25}$$

Substitutes the above expression in step 1 and integrates the integral in step 1 to get the following expression:

$$x\cot^{-1}\left(\frac{5}{x}\right) - 5\int\frac{x}{x^2+25}dx$$

Completes integrating the above expression to get:

$$x\cot^{-1}\left(\frac{5}{x}\right) - \frac{5}{2}\log|x^2+25| + C$$

where C is an arbitrary constant.

(Award full marks even if modulus is not used in log function as $(x^2 + 25)$ is always positive.)

2. Substitutes $\sqrt{(x^2+4)} + x$ as t and finds x as:

$$\sqrt{(x^2+4)} = t - x$$

Squares both sides to get:

$$2tx = t^2 - 4$$

$$x = \frac{t}{2} - \frac{2}{t}$$

Finds dx as $\left(\frac{1}{2} + \frac{2}{t^2}\right)dt$.

Rewrites the integral as:

$$\int\sqrt{t}\left(\frac{1}{2}+\frac{2}{t^2}\right)dt$$

$$\Rightarrow \frac{1}{2}\int t^{\frac{1}{2}}dt + 2\int t^{\frac{-3}{2}}dt$$

Integrates the above expression as:

$$\frac{(t)^{\frac{3}{2}}}{3} - 4(t)^{-\frac{1}{2}} + C$$

where C is an arbitrary constant.

Substitutes t as $\sqrt{(x^2+4)} + x$ to get:

$$\frac{(\sqrt{x^2+4}+x)^{\frac{3}{2}}}{3} - \frac{4}{(\sqrt{x^2+4}+x)^{\frac{1}{2}}} + C$$