CHAPTER 9

Differential Equation

Level - 1 CORE SUBJECTIVE QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ) (1 Marks)

1. **Option (B) is correct.**

Explanation: A differential equation of the form $\frac{dy}{dx}=f(x,y)$ is said to be homogeneous, if $f(x, y)$ is a homogeneous function of degree 0.

Now,

$$x^n\frac{dy}{dx}=y\left(\log_e\frac{y}{x}+\log_e e\right)\Rightarrow\frac{dy}{dx}=\frac{y}{x^n}\left(\log_e e\cdot\left(\frac{y}{x}\right)\right)$$

$= f(x, y)$; (Let). $f(x, y)$ will be a homogeneous function of degree 0, if $n = 1$.

2. **Option (B) is correct.**

Explanation: The given differential equation $e^{y'} = x$

$$\Rightarrow\frac{dy}{dx}=\log x$$

$$dy = \log x\, dx$$

$$\Rightarrow \int dy=\int\log x\, dx$$

$$y = x\log x - x + c$$

3. **Option (B) is correct.**

Explanation: $x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^5=0$

Order: The highest order derivative present is $\frac{d^2y}{dx^2}$, so order = 2.

Degree: The highest power of the highest order derivative is 3.

4. **Option (C) is correct.**

Explanation: $\frac{dy}{dx}+y\tan x=2x+x^2\tan x$

The standard form of the linear differential equation is:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where $P(x) = \tan x$.

The integrating factor (IF) is given by:

$$e^{\int P(x)dx}=e^{\int\tan x dx}=e^{\log|\sec x|}=\sec x$$

5. **Option (C) is correct.**

Explanation: Given

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\frac{d^2y}{dx^2}$$

Order: The highest order derivative present is $\frac{d^2y}{dx^2}$, so the order is 2.

Degree: The equation is already polynomial in $\frac{d^2y}{dx^2}$, and the power of $\frac{d^2y}{dx^2}$ is 1.

So, degree = 1

6. **Option (C) is correct.**

Explanation: The standard linear form is:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$\frac{dy}{dx}-\frac{y}{x}=x^3-3$$

Here, $P(x)=-\frac{1}{x}\cdot$

The integrating factor (IF) is:

$$e^{\int P(x)dx}=e^{\int-\frac{1}{x}dx}=e^{-\log x}=x^{-1}$$

7. **Option (B) is correct.**

Explanation: Given,

$$\frac{dy}{dx}=\frac{1}{\log y}$$

$$\Rightarrow (\log y)dy = dx$$

Integrate both sides:

$$\int\log y.1dy=\int dx$$

Usingintegrationbyparts $\int u.v dx=u\int v dx-\int\left(u'\int v dx\right)dx$

$$\log y.y-\int\frac{1}{y}.y.dy$$

$$y\log y-y=x+c$$

8. **Option (D) is correct.**

Explanation: $\left(1-x^2\right)\frac{dy}{dx}+xy=ax,\ -1<x<1$

The equation is in the standard form:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where $P(x)=\frac{x}{1-x^2}$

The integrating factor (IF) is:

$$e^{\int P(x)dx}=e^{\int\frac{x}{1-x^2}dx}$$

Let $u=1-x^2$, then $du=-2xdx$, so:

$$\int\frac{x}{1-x^2}dx=-\frac{1}{2}\log\left|1-x^2\right|$$

$$IF=e^{-\frac{1}{2}\log\left|1-x^2\right|}=(1-x^2)^{-1/2}=\frac{1}{\sqrt{1-x^2}}$$

9. **Option (A) is correct.**

Explanation: Solution:

The differential equation $\frac{dy}{dx}=F(x,y)$ is homogenous differential equation if $F(\lambda x,\lambda y)=F(x,y)$.

Examining the above options, (B), (C) and (D) are homogeneous differential equation as below.

(B) $F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}=\frac{y}{x}=F(x,y)$

(C) $F(\lambda x,\lambda y)=\frac{(\lambda x)^2+(\lambda y)^2}{\lambda x\lambda y}=\frac{\lambda^2x^2+\lambda^2y^2}{\lambda^2xy}$

$$=\frac{x^2+y^2}{xy}=F(x,y)$$

(D) $F(\lambda x,\lambda y)=\cos^2\left(\frac{\lambda y}{\lambda x}\right)=\cos^2\left(\frac{y}{x}\right)=F(x,y)$

Now,

(A) $F(\lambda x,\lambda y)=\cos\lambda x-\sin\left(\frac{\lambda y}{\lambda x}\right)=\cos\lambda x-\sin\left(\frac{y}{x}\right)$

$\neq F(x,y)$, so (A) is not a homogeneous differential equation

10. **Option (D) is correct.**

Explanation: $\left(y''\right)^2+\left(y'\right)^3=x\sin\left(y'\right)$

Order: The highest order derivative is y'', so order = 2.

Degree: The equation must be polynomial in the highest derivative.

Since $x\sin(y')$ contains a trigonometric function of y', it is not a polynomial.

Hence, the degree is not defined.

11. **Option (B) is correct.**

Explanation: $\left(y''\right)^2+\log\left(y'\right)=x^5$

Order: The highest order derivative is y'', so order = 2.

Degree: The equation must be polynomial in the highest derivative.

Since $\log(y')$ is a non-polynomial function, the degree is not defined.

12. **Option (C) is correct.**

Explanation: $x\log x\frac{dy}{dx}+y=2\log x$

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2\log x}{x\log x}$$

This is in the standard form of a first-order linear differential equation:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

So, this is a first-order linear differential equation.

13. **Option (A) is correct.**

Explanation: Given

$$xdy+ydx=0$$

$$\Rightarrow\quad\frac{dy}{dx}=-\frac{y}{x}$$

This is a separable differential equation:

$$\frac{dy}{y}=\frac{-dx}{x}$$

Integrating both sides:

$$\ln|y|=-\ln|x|+\ln c$$
$$\ln|y|+\ln|x|=\ln c$$
$$\ln|xy|=\ln c$$
$$xy=c$$

14. **Option (D) is correct.**

Explanation: Given, Differential Eqn. $(x+2y^2)\frac{dy}{dx}=y$

$$\Rightarrow\quad\frac{dy}{dx}=\frac{y}{x+2y^2}$$

$$\Rightarrow\quad\frac{dx}{dy}=\frac{x+2y^2}{y}$$

$$\Rightarrow\quad\frac{dx}{dy}=\frac{x}{y}+2y$$

$$\Rightarrow\quad\frac{dx}{dy}-\frac{x}{y}=2y$$

We know that, $\frac{dx}{dy}+Px=Q, P=\frac{-1}{y}, Q=2y$

$$\text{I.F.}=e^{\int P dy}$$

$$= e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

$$\text{I.F.} = \frac{1}{y}$$

15. Option (C) is correct.

Explanation: The equation contains the given initial condition $y(0) = 0$, meaning that the particular solution is uniquely determined (no arbitrary constants remain).

16. Option (B) is correct.

Explanation:

$$\frac{dy}{dx} + \frac{2}{x}y = 0, \quad x \neq 0$$

The standard form is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Here,

$$P(x) = \frac{2}{x}$$

$$\text{IF} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

17. Option (A) is correct.

Explanation: The order of a differential equation is the highest derivative present.

Here, the highest derivative is $\frac{d^4y}{dx^4}$ (4^{th} derivative).

18. Option (A) is correct.

Explanation:

$$\frac{dy}{dx} = e^{x+y}$$

$$e^{-y}dy = e^x dx$$

Integrate both sides:

$$\int e^{-y}dy = \int e^x dx$$

$$-e^{-y} + c = e^x$$

$$e^{-y} + e^x = c$$

19. Option (C) is correct.

Explanation: The highest derivative of a differential equation is $\frac{d^4y}{dx^4}$. So, the order is 4.

20. Option (D) is correct.

Explanation: Given the differential equation:

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{1}{x^2-1}$$

Dividing by $x^2 - 1$, we get:

$$\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2}$$

Here, $P(x) = \frac{2x}{x^2-1}$

The integrating factor is:

$$IF = e^{\int \frac{2x}{x^2-1}dx} = e^{\ln|x^2-1|} = x^2 - 1$$

21. Option (C) is correct.

Explanation: Given the differential equation:

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides:

$$\ln|y| = -\ln|x| + C$$

$$\ln|y| + \ln|x| = C$$

$$\ln|yx| = C$$

$$yx = e^C = C'$$

$$y = \frac{C'}{x}$$

Using given condition $x = 1, y = 1$,

$$1 = \frac{C'}{1} \Rightarrow C' = 1$$

Thus,

$$y = \frac{1}{x}$$

22. Option (B) is correct.

Explanation: The general solution of

$$\frac{dy}{dx} + y = 0$$

On comparing

$$\frac{dy}{dx} + P(x)y = 0$$

where $P(x) = 1$. The integrating factor (IF) is:

$$IF = e^{\int P(x)dx} = e^{\int 1 dx} = e^x$$

Multiplying by e^x:

$$e^x\frac{dy}{dx} + e^x y = 0$$

$$\frac{d}{dx}(ye^x) = 0$$

Integrating both sides:

$$ye^x = C$$

$$y = Ce^{-x}$$

Since it contains one arbitrary constant C.

23. Option (C) is correct.

Explanation:

(A) Order = 3, Degree = 1

(B) Order = 3, Degree is not defined due to sine function

(C) Order = 2, Degree = 2

(D) Order = 3, Degree = 1

24. Option (A) is correct.

Explanation: Given Differential Equation (i):

$$xe^{x/y}dx - ye^{3x/y}dy = 0$$

We have, $y = bx$

$$\frac{y}{x} = b \Rightarrow \frac{x}{y} = \frac{1}{b}$$

Differentiating both sides,

$$\frac{dy}{dx} = b + x\frac{db}{dx}$$

Now, substituting $y = bx$ into the given equation:

$$xe^{x/y}dx - ye^{3x/y}dy = 0$$

$$xe^{1/b}dx - bxe^{3/b}dy = 0$$

Using $dy = bdx + xdb$, we substitute:

$$xe^{1/b}dx - bxe^{3/b}(bdx + xdb) = 0$$

$$xe^{1/b}dx - b^2xe^{3/b}dx - bx^2e^{3/b}db = 0$$

$$(xe^{1/b} - b^2xe^{3/b})dx = bx^2e^{3/b}db$$

Dividing both sides by x :

$$(e^{1/b} - b^2e^{3/b})dx = bxe^{3/b}db$$

$$\frac{dx}{x} = \frac{be^{3/b}db}{e^{1/b} - b^2e^{3/b}}$$

This is now in variable separable form, confirming that equation (i) is separable when substituting $y = bx$.

Equation (ii): $(2x+1)\frac{dy}{dx} = 3 - 2y$

Substituting $y = bx$

Since $y = bx$, we differentiate:

$$\frac{dy}{dx} = b + x\frac{db}{dx}$$

Substituting into the given equation:

$$(2x+1)\left(b + x\frac{db}{dx}\right) = 3 - 2bx$$

$$(2x+1)b + (2x+1)x\frac{db}{dx} = 3 - 2bx$$

$$(2x+1)x\frac{db}{dx} = 3 - 2bx - (2x+1)b$$

$$(2x+1)x\frac{db}{dx} = 3 - 2bx - 2bx - b$$

$$(2x+1)x\frac{db}{dx} = 3 - 4bx - b$$

This equation is not separable in terms of x and b, so equation (ii) is not separable.

Equation (iii): $\frac{dy}{dx} = \sin x - \cos y$

Substituting $y = bx$

Differentiating:

$$\frac{dy}{dx} = b + x\frac{db}{dx}$$

Substituting into the given equation:

$$b + x\frac{db}{dx} = \sin x - \cos(bx)$$

$$x\frac{db}{dx} = \sin x - \cos(bx) - b$$

This equation is not separable in terms of x and b, so Equation (iii) is also not separable.

25. Option (B) is correct.

Explanation: Given,

$$\frac{dy}{dx} = e^{x-y}$$

$$\Rightarrow \qquad e^y dy = e^x dx$$

Integrating:

$$\int e^y dy = \int e^x dx$$

$$e^y + C = e^x$$

$$e^x - e^y = C$$

26. Option (A) is correct.

Explanation: Given:

$$\left(\frac{d^3y}{dx^3}\right)^2 + 3x\left(\frac{d^2y}{dx^2}\right)^4 = \log x$$

Order = 3 (highest derivative d^3y/dx^3)

Degree = 2 (power of the highest order derivative after removing radicals/fractions)

Sum = 3 + 2 = 5

27. Option (C) is correct.

Explanation: Given:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y$$

Order = 2

Degree =1 (since highest order derivative appears with power 1)

Sum = 2 + 1 = 3

28. Option (D) is correct.

Explanation: Given:

$$xdy - (1+x^2)dx = dx$$

$$x\frac{dy}{dx} = 1 + x^2 + 1$$

$$x\frac{dy}{dx} = 2 + x^2$$

$$dy = \left(\frac{2}{x} + x\right)dx$$

Integrating

$$y = 2\ln|x| + \frac{x^2}{2} + C$$

29. Option (C) is correct.

Explanation: Given:

$$\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$$

Since $\cos\left(\frac{dy}{dx}\right)$ is a non-polynomial function of the derivative, degree is not defined.

30. Option (D) is correct.

Explanation: Given, $(1-y^2)\frac{dx}{dy}+yx=ay$

$$\frac{dx}{dy}+\frac{y}{1-y^2}x=\frac{ay}{1-y^2}$$

The integrating factor (IF)

$$IF=e^{\int\frac{y}{1-y^2}dy}$$

Let $u = 1 - y^2$, then $du = -2ydy$, so:

$$\int\frac{y}{1-y^2}dy=-\frac{1}{2}\int\frac{du}{u}=-\frac{1}{2}\ln\left|1-y^2\right|$$

Thus,

$$IF=e^{-\frac{1}{2}\ln\left|1-y^2\right|}=\left|1-y^2\right|^{-\frac{1}{2}}$$

$$IF=\frac{1}{\sqrt{1-y^2}}$$

31. **Option (C) is correct.**

Explanation: Given:

$$\frac{dx}{x}+\frac{dy}{y}=0$$

Separating variables:

$$\int\frac{dx}{x}=-\int\frac{dy}{y}$$

$$\ln|x|=-\ln|y|+C_1$$

$$\ln|x|+\ln|y|=C_1$$

$$\ln xy=C_1$$

$$xy=e^{C_1}=C$$

$$xy=C$$

32. **Option (B) is correct.**

Explanation: Given:

$$\frac{d^2y}{dx^2}\sin y+\left(\frac{dy}{dx}\right)^3\cos y=\sqrt{y}$$

Order = 2 (highest derivative is $\frac{d^2y}{dx^2}$)

Degree = 1

Product = 2 × 1 = 2

33. **Option (D) is correct.**

Explanation: Given:

$$x\frac{dy}{dx}-y=2x^2$$

$$\frac{dy}{dx}-\frac{y}{x}=2x$$

Comparing with $\frac{dy}{dx}+P(x)y=Q(x)$, we have

$P(x) = -\frac{1}{x}$. The integrating factor (IF) is:

$$IF=e^{\int(-1/x)dx}=e^{-\ln x}=\frac{1}{x}$$

34. **Option (D) is correct.**

Explanation: Given:

$$\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3=x\sin\left(\frac{dy}{dx}\right)$$

Order =2 (highest derivative is $\frac{d^2y}{dx^2}$)

Degree = not defined (since sin (dy/dx) is non-polynomial)

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (D) is correct.**

Explanation:

Statement 1: Statement 1 is false as the given equation is not a differential equation, it is actually the general solution of differential equation.

Statement 2: Given differential equation:

Order of a differential equation is the highest derivative present. Here, the highest derivative is $\frac{dy}{dx}$, so the order is 1.

The degree is the power of the highest-order derivative when the equation is expressed in polynomial form. Here, the degree is 1.

Since the order is 1, not 0, Statement 2 is false.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. Given differential equation can be written as

$$x^2+\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}=\left(\frac{dy}{dx}\right)^2+1$$

i.e. $x^2+2x\frac{dy}{dx}=1$; Order = 1, degree = 1

Sum of order and degree = 1 + 1 = 2

2. Given differential equation is

$$\frac{dy}{dx}=e^{x-y}$$

i.e. $e^y dy = e^x\, dx$

Integrating both sides, we get

$$e^y = e^x + C$$

3. Given differential equation can be written as

$$2xy\left(\frac{dy}{dx}\right)^2+y^2\frac{dy}{dx}+y=0;\ \text{Order} = 1, \text{Degree} = 2$$

Order × degree = 1 × 2 = 2

4. Given differential equation can be written as

$$\frac{dy}{y^2}=2dx$$

$$\int \frac{dy}{y^2} = 2\int dx$$

$$\frac{-1}{y} = 2x + C$$

putting $x = 1, y = 1$ gives $C = -3$

$$\therefore \quad -\frac{1}{y} = 2x - 3$$

or $$y = \frac{-1}{2x-3}$$

5. $$\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x - e^{-x}}$$

$$\frac{dy}{dx} = \frac{3e^{2x}(1+e^{2x})}{e^x + \frac{1}{e^x}}$$

$$= \frac{3e^{2x}(1+e^{2x})}{e^{2x}+1} \times e^x$$

$$dy = \int 3e^{3x} dx \Rightarrow y = e^{3x} + C$$

6. $$\log\left(\frac{dy}{dx}\right) = ax + by$$

$$\frac{dy}{dx} = e^{ax + by}$$

$$\frac{dy}{dx} = e^{ax}.e^{by}$$

$$\int \frac{dy}{e^{by}} = \int e^{ax}\, dx$$

$$\frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + C$$

7. $$e^{dy/dx} = x^2$$

$$\frac{dy}{dx} = \log x^2 = 2 \log x$$

(Taking by both sides)

$$\int dy = 2\int \log x\, dx$$

$$y = 2[\log x \times x - \int 1\, dx]$$

$$y = 2[x \log x - x] + C$$

8. Given differential equation can be written as $\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$

Integrating, $\log|\tan x| + \log|\tan y| = \log C$

$\tan x.\tan y = C$

9. $$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\therefore \quad \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \quad \frac{dy}{e-y} = dx\,(e^x + x^2)$$

Integrating both sides, we get,

$$\int \frac{dy}{e^{-y}} = \int (x^2 + e^x) dx$$

$$e^y = \frac{x^3}{3} + e^x + C$$

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. $$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{2x}\right)$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = v + \cos^2\left(\frac{v}{2}\right)$$

$$\Rightarrow \quad \int \sec^2\left(\frac{v}{2}\right)dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad 2\tan\left(\frac{v}{2}\right) = \log|x| + C$$

$$\Rightarrow \quad 2\tan\left(\frac{y}{2x}\right) = \log|x| + C$$

At $x = 1, y = \frac{\pi}{2}$, we get

$$2\tan\frac{\pi}{4} = \log 1 + C \Rightarrow C = 2$$

$$\therefore \quad 2\tan\left(\frac{y}{2x}\right) = \log|x| + 2$$

2. Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2xvx}$$

$$\int \frac{1}{x}dx = \int \frac{2v}{1-v^2}dv$$

$$\Rightarrow \quad \log|x| = -\log|1-v^2| + \log C$$

$$\log|x(1-v^2)| = \log C$$

$$\Rightarrow \quad x\left(1 - \frac{y^2}{x^2}\right) = C \text{ or } x^2 - y^2 = Cx$$

3. $$(\tan^{-1}y - x)\, dy = (1 + y^2)\, dx$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Solution is:

$$x.e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y} dy$$

$$\Rightarrow \quad xe^{\tan^{-1}y} = (\tan^{-1}y)e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

$$\Rightarrow x = \tan^{-1}y - 1 + C\,e^{-\tan^{-1}y}$$

4. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2} = \frac{y}{x} + \frac{y^2}{2x^2}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

The equation becomes

$$x\frac{dv}{dx} = \frac{1}{2}v^2$$

$$\Rightarrow \quad \frac{dv}{v^2} = \frac{1}{2} \times \frac{dx}{x}$$

Integrating both sides, we get

$$\frac{-1}{v} = \frac{1}{2}\log|x| + C$$

$$\Rightarrow \quad -\frac{x}{y} = \frac{1}{2}\log|x| + C$$

$y = 2, x = 1$ gives $C = -\frac{1}{2}$

The particular solution is

$$-\frac{x}{y} = \frac{1}{2}\log|x| - \frac{1}{2}$$

or, $$y = \frac{2x}{1 - \log|x|}$$

5. Given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

Integrating factor $= e^{\int \frac{-1}{y}dy} = \frac{1}{y}$

Solution is $x\frac{1}{y} = \int 2dy$

$$\Rightarrow \quad \frac{x}{y} = 2y + C$$

$$\Rightarrow \quad x = 2y^2 + Cy$$

6. $$\frac{dy}{dx} = y \cot 2x$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \cot 2x\, dx$$

$$\Rightarrow \quad \log|y| = \frac{1}{2}\log|\sin 2x| + \log c$$

$$y = c\sqrt{\sin 2x}$$

when $y\left(\frac{\pi}{4}\right) = 2$, gives $c = 2$

$\therefore y = 2\sqrt{\sin 2x}$ is the required particular solution of given D.E.

7. $$\frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x} = f\left(\frac{y}{x}\right)$$

so, its a homogeneous differential equation

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, $v + x\frac{dv}{dx} = e^v + v$

$$\Rightarrow \quad \int e^{-v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad -e^{-v} = \log|x| + c$$

$$\Rightarrow \quad -e^{\frac{-y}{x}} = \log|x| + c$$

Now, $x = 1, y = 1$, gives $c = -e^{-1}$

Thus, $\log|x| + e^{\frac{-y}{x}} = e^{-1}$

8. Given differential equation is a linear order differential equation with:

$$P = -2x,\ Q = 3x^2e^{x^2}$$

Integrating Factor $= e^{\int -2x dx} = e^{-x^2}$

The general solution is:

$$y \cdot e^{-x^2} = \int e^{-x^2}.3x^2e^{x^2}dx + C$$

$$\Rightarrow \quad y.e^{-x^2} = x^3 + C$$

Putting $x = 0, y = 5$, we get, $C = 5$

$\therefore$ The required solution is:

$$y \cdot e^{-x^2} = x^3 + 5 \text{ or } y = (x^3 + 5)e^{x^2}$$

9. $x^2dy + y(x + y)dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} - \left(\frac{y}{x}\right)^2$$

Putting $\frac{y}{x} = v \Rightarrow y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = -v - v^2,$$

separating the variable and integrating

$$\int \frac{1}{v^2 + 2v}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{1}{(v+1)^2 - 1}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = \log\left|\frac{C}{x}\right|$$

The solution of the differential equation is,

$$\left|\frac{y}{y+2x}\right| = \frac{C^2}{x^2}$$

or $\quad x^2y = k(y + 2x) \quad$ [where $k = C^2$]

10. The given differential equation can be written as,

$$\frac{dy}{dx} - \frac{1}{x}y = \log x$$

$$\therefore \; P = -\frac{1}{x}, Q = \log x$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

The solution of the differential equation is:

$$y \cdot \frac{1}{x} = \int \log x \cdot \frac{1}{x}\, dx$$

$$\Rightarrow \frac{y}{x} = \frac{(\log x)^2}{2} + c, \text{ or } y = \frac{1}{2}x.(\log x)^2 + cx$$

11. $$\frac{dy}{dx} + y.\cot x = 4\, x \text{ cosec } x$$

$$\text{I.F.} = e^{\int \cot x dx} = e^{\log(\sin x)} = \sin x$$

Solution is given by:

$$y.(\sin x) = \int (4x.\text{cosec } x).\sin x\, dx$$

$$\Rightarrow \quad y.\sin x = \int 4x\, dx$$

$$\Rightarrow \quad y.\sin x = 2x^2 + C$$

Now, $y = 0, x = \frac{\pi}{2}$ gives $C = -\frac{\pi^2}{2}$

Required solution is: $y.\sin x = 2x^2 - \frac{\pi^2}{2}$

12. $$xdy - ydx = \sqrt{x^2 + y^2}dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad ...(1)$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(2)

From (1) and (2), we get

$$x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log|v + \sqrt{1+v^2}| = \log|x| + \log C$$

$$\log\left|\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\right| = \log|x| + \log C$$

or $\quad y + \sqrt{x^2 + y^2} = Cx^2$

13. Integrating factor $= e^{\int \sec^2 x\, dx} = e^{\tan x}$

Solution is $ye^{\tan x} = \int \tan x \sec^2 x\, e^{\tan x} dx + C$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore\; y\, e^{\tan x} = \int e^t t\, dt \Rightarrow y.e^{\tan x} = e^t(t-1) + C$

or $ye^{\tan x} = e^{\tan x}(\tan x - 1) + C$

$y(0) = 0$ gives $C = 1$

Particular solution is $ye^{\tan x} = e^{\tan x}(\tan x - 1) + 1$

or $y = \tan x - 1 + e^{-\tan x}$

14. Integrating factor is $e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

Solution is $y \times \sin x = \int \cos^2 x \sin x dx + C$

$$\Rightarrow \quad y \sin x = -\frac{\cos^3 x}{3} + C$$

$x = \frac{\pi}{2}, y = 0 \Rightarrow C = 0$

$\therefore$ Particular solution is $y \sin x = -\frac{\cos^3 x}{3}$

or $y = -\frac{\cos^3 x}{3}.\text{cosec } x$

15. Let $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

Substituting in the given differential equation, we get

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v+1}$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{(e^v + v)}{e^v + 1}$$

$$\Rightarrow \quad \frac{e^v + 1}{e^v + v}dv = -\frac{dy}{y}$$

Integrating, we get

$$\log|e^v + v| = -\log|y| + \log C$$

$$\Rightarrow \quad e^{x/y} + \frac{x}{y} = \frac{C}{y}$$

or $\quad ye^{x/y} + x = C$

16. Integrating factor $= e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Solution is $y.\frac{1}{x^2} = \int \frac{1}{x^2} \cdot \sin\frac{1}{x}dx + C$

Let $\frac{1}{x} = t, -\frac{1}{x^2}dx = dt$

$$\therefore \quad y \cdot \frac{1}{x^2} = -\int \sin t\, dt + C \Rightarrow \frac{y}{x^2} = \cos t + C$$

$$\therefore \quad y\cdot\frac{1}{x^2} = \cos\frac{1}{x}+C$$

17. Given differential equation becomes

$$\frac{dy}{dx} = 2\sin x\cos y$$

$$\Rightarrow \quad \sec y\,dy = 2\sin x\,dx$$

On integrating, we get

$$\log|\sec y+\tan y| = -2\cos x + C$$

$x=\frac{\pi}{4}, y=0$ gives $C=\sqrt{2}$

$\therefore$ Particular solution is

$$\log|\sec y+\tan y| = -2\cos x+\sqrt{2}$$

18. Rewriting the given differential equation as:

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y = \frac{\sqrt{x^2+4}}{1+x^2}$$

$$\text{Integrating factor } = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

$\therefore$ solution of the differntial equation is

$$y(1+x^2) = \int\frac{\sqrt{x^2+4}}{1+x^2}\cdot(1+x^2)dx+c$$

$$= \int\sqrt{x^2+4}\,dx+c$$

$$\therefore \quad y(1+x^2) = \frac{x\sqrt{x^2+4}}{2}+2\log|x+\sqrt{x^2+4}|+c$$

19. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{y^2}{xy-x^2} = \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x}-1}$$

Put $y=ux \Rightarrow \frac{dy}{dx}=u+x\frac{du}{dx}$

$$\therefore \quad u+x\frac{du}{dx} = \frac{u^2}{u-1}$$

$$\Rightarrow \quad x\frac{du}{dx} = \frac{u}{u-1}$$

Separating the variables and integrating

$$\int\left(1-\frac{1}{u}\right)du = \int\frac{dx}{x}$$

$$\Rightarrow \quad u-\log u = \log|x|+c$$

$$\Rightarrow \quad \frac{y}{x}-\log\frac{y}{x} = \log|x|+c$$

20. Given differential equation is

$$2xy\frac{dy}{dx}+y^2 = 2y(1+x^2)$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{2x} = \frac{1}{x}+x$$

$$\text{Integrating factor } = e^{\int\frac{1}{2x}dx} = e^{\log\sqrt{x}} = \sqrt{x}$$

Solution is given by $y\sqrt{x} = \int\left(\frac{1}{\sqrt{x}}+x^{\frac{3}{2}}\right)dx + C$

$$\Rightarrow y\sqrt{x} = 2\sqrt{x}+\frac{2x^{\frac{5}{2}}}{5}+C \text{ or } y = 2+\frac{2x^2}{5}+\frac{C}{\sqrt{x}}$$

21. Given differential equation is $\frac{dy}{dx}=\frac{y}{x}-e^{\frac{y}{x}}$

Let $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

The given equation becomes

$$v+x\frac{dv}{dx} = v-e^v$$

$$\Rightarrow \quad -e^{-v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$e^{-v} = \log|x|+C$$

$$\Rightarrow \quad e^{-\frac{y}{x}} = \log|x|+C$$

22. $$\frac{dy}{dx} = \frac{x+y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1+\frac{y}{x}$$

Let $\frac{y}{x}=v$. Then $x\frac{dv}{dx}+v=\frac{dy}{dx}$

So, differential equation becomes:

$$x\frac{dv}{dx}+v = 1+v$$

$$\Rightarrow \quad dv = \frac{dx}{x}$$

$$\Rightarrow \quad v = \log|x|+c$$

$$\Rightarrow \quad y = x\log|x|+cx$$

$\Rightarrow x=1, y=0 \Rightarrow c=0$, $\therefore$ particular solution is

$$y = x\log|x|$$

23. The given D.E. is

$$\frac{\sec^2 y}{\tan y}dy = -\frac{e^x}{1-e^x}dx$$

Integrating both sides,

$$\Rightarrow \quad \log|\tan y| = \log|1-e^x|+\log C$$

$$\Rightarrow \quad \tan y = C(1-e^x)$$

24. The given differential equation can be written as,

$$\frac{dy}{dx} = \frac{y}{x}-\tan\frac{y}{x}$$

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \quad v+x\frac{dv}{dx} = v-\tan v$$

$\Rightarrow \quad \int \frac{dv}{\tan v} = -\int \frac{dx}{x}$

$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C$

$\Rightarrow \sin\frac{y}{x} = \frac{C}{x} \Rightarrow x\sin\frac{y}{x} = C$

25. The given differential equation can be written as

$$\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

The integrating factor, I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ The solution is:

$$y \times x = \int (x\cos x + \sin x)dx$$

$$= x\sin x - \int \sin x dx + \int \sin x dx + C$$

$y \times x = x\sin x + C \Rightarrow y = \sin x + \frac{C}{x}$

Put, $x = \frac{\pi}{2}, y = 1 \Rightarrow C = 0$

$\therefore$ The particular solution is $y = \sin x$.

26. $$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} \quad ...(1)$$

Put $\frac{y}{x} = v$ i.e. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (1) gives $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^3}{1+v^2}$

$\Rightarrow \quad \int \frac{1+v^2}{v^3} dv = -\int \frac{dx}{x}$

$\Rightarrow \quad \frac{-1}{2v^2} + \log|v| = -\log|x| + \log c$

Putting $v = \frac{y}{x}$ and simplifying gives

$$-\frac{1}{2}\frac{x^2}{y^2} + \log\left|\frac{y}{x}\right| = -\log|x| + \log c$$

$$-\frac{x^2}{2y^2} = \log\left|\frac{c}{y}\right|$$

Now, $x = 0, y = 1$ gives $c = 1$

Required solution is: $\frac{x^2}{2y^2} = \log|y|$

27. Given differential can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{1}{(1+x^2)^2}$$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2$$

Solution is given by: $y.(1+x^2) = \int \frac{1}{1+x^2} dx$

$\Rightarrow \quad y.(1 + x^2) = \tan^{-1}x + C$

Now, $x = 1, y = 0$ gives $C = -\frac{\pi}{4}$

Particular solution is $(1+x^2) = \tan^{-1}x - \frac{\pi}{4}$

28. Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, given differential equation can be written as:

$$v + x\frac{dv}{dx} = \frac{1+v^2}{v}$$

Gives $\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{v}$

$$v\,dv = \frac{dx}{x}$$

On integrating, we get

$$\frac{v^2}{2} = \log|x| + C$$

$$\frac{y^2}{2x^2} = \log|x| + C$$

29. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x} \quad ...(1)$$

Let $y = vx \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes $\not{v} + x\frac{dv}{dx} = \not{v} - \cos^2 v$

$$\sec^2 v\,dv = -\frac{dx}{x}$$

Integrating both sides we get

$$\tan v = -\log|x| + c$$

$$\tan\frac{y}{x} = -\log|x| + c$$

At $x = 1, y = \frac{\pi}{4} \Rightarrow c = 1$

$\therefore$ Particular solution is $\tan\frac{y}{x} = -\log|x| + 1$

30. Given differential equation can be written as

$$\frac{dx}{dy} = \frac{x}{y + 3x^2}$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y + 3x^2}{x}$

$$x\frac{dy}{dx} - y = 3x^2$$

or $\quad \frac{dy}{dx} - \frac{1}{x}y = 3x$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

Solution is $y \cdot \frac{1}{x} = \int 3x \frac{1}{x} dx + C$

$$\frac{y}{x} = 3x + C$$

$x = 1$ $y = 1$ gives $C = -2$

Particular solution is $\frac{y}{x} = 3x - 2$ or $y = 3x^2 - 2x$

31. $x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin\frac{y}{x}$$

Let $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, given above differential equation becomes:

$$v + x\frac{dv}{dx} = v - \sin v$$

$$\text{cosec } v \, dv = -\frac{dx}{x}$$

$\Rightarrow \log |\text{cosec } v - \cot v| + \log |x| = \log C$

$$\Rightarrow x\left(\text{cosec } \frac{y}{x} - \cot\frac{y}{x}\right) = C$$

32. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{e^{y/x}\left(\frac{y}{x} - 1\right)}{1 + e^{y/x}}$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{e^v(v-1)}{1+e^v}$$

$$x\frac{dv}{dx} = \frac{-e^v - v}{1+e^v}$$

$$\int \frac{(e^v + 1)}{e^v + v} dv = -\int \frac{dx}{x}$$

$$\log |e^v + v| = -\log |x| + \log C$$

$$e^v + v = \frac{C}{x}$$

$$e^{y/x} + \frac{y}{x} = \frac{C}{x}$$

or $\quad xe^{y/x} + y = C$

33. The given equation can be written as:

$$\frac{dy}{dx} - \frac{y}{x} = \left(\frac{x^2}{x}\right)e^x = xe^x$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

Solution is $y \times \frac{1}{x} = \int e^x \, dx + C$

$$\frac{y}{x} = e^x + C$$

$$y = xe^x + Cx$$

At $x = 1, y = 0,$ $C = -e$

Particular solution is

$$y = xe^x - ex$$

34. Given, $\quad x\frac{dy}{dx} = y[\log y - \log x + 1]$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x} + 1\right]$$

Put $\quad y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Therefore, $\quad v + x\frac{dv}{dx} = v[\log v + 1]$

$$x\frac{dv}{dx} = v \log v$$

$$\int \frac{dv}{v \log v} = \int \frac{1}{x} dx$$

$$\log |\log v| = \log |x| + \log C$$

$$\log\left(\frac{y}{x}\right) = Cx$$

35. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{2y^4 + 5x^3y}{xy^3 + x^4}$$

$$\frac{dy}{dx} = \frac{2\left(\frac{y}{x}\right)^4 + 5\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^3 + 1} \quad \text{...(i)}$$

Let, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (1) becomes $v + x\frac{dv}{dx} = \frac{2v^4 + 5v}{v^3 + 1}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^4 + 4v}{v^3 + 1}$$

$$\frac{v^3 + 1}{v^4 + 4v} dv = \frac{dx}{x}$$

$$\int \frac{4v^3 + 4}{v^4 + 4v} dv = 4\int \frac{dx}{x}$$

$$\log |v^4 + 4v| = \log(x)^4 + \log C$$

$$\log\left|\frac{y^4 + 4yx^3}{x^4}\right| = \log Cx^4$$

$$y^4 + 4yx^3 = Cx^8$$

36. Given differential equation can be written as

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{-1}{x(1+x^2)}$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Solution is $y \times x = \int \frac{-1}{1+x^2}dx + C$

$$\Rightarrow \quad xy = -\tan^{-1} x + C$$

Now $y(1) = 0 \Rightarrow C = \frac{\pi}{4}$

$\therefore$ Particular solution is $xy = \frac{\pi}{4} - \tan^{-1} x$

37. $(y - \sin^2 x)\, dx + \tan x\, dy = 0$

$$\frac{dy}{dx} + \frac{y}{\tan x} = \frac{\sin^2 x}{\tan x}$$

$$\frac{dy}{dx} + (\cot x)y = \sin x \cos x$$

$$\text{I.F.} = e^{\int \cot x\, dx} = e^{\log(\sin x)} = \sin x$$

Solution is given by

$$y \sin x = \int \sin^2 x \cos x\, dx$$

$$y \sin x = \int t^2 dt + C$$

$[\because \sin x = t \therefore \cos x\, dx = dt]$

$$y \sin x = \frac{t^3}{3} + C$$

$$y \sin x = \frac{\sin^3 x}{3} + C$$

38. Given, $\quad \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^3} \quad \text{...(1)}$$

Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

$$\int \frac{1+v^3}{v^4} dv = -\int \frac{dx}{x}$$

$$\int \frac{1}{v^4} dv + \int \frac{1}{v} dv = -\log|x| + C$$

$$\frac{1}{-3v^3} + \log|v| = -\log|x| + C$$

$$\frac{-x^3}{3y^3} + \log\left|\frac{y}{x}\right| = -\log|x| + C$$

or $$\frac{-x^3}{3y^3} + \log|y| = C$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1. Substitutes $\sin x$ as t and finds $\frac{dx}{dt} = \frac{1}{\cos x}$.

Writes that from the given equation and $\frac{dx}{dt} = \frac{1}{\cos x}$ we can find:

$$\frac{dy}{dt} = \frac{y + \sqrt{t^2 - y^2}}{t}$$

where $t = \sin x$.

Substitutes $y = vt$ and gets $\frac{dy}{dt} = v + t\frac{dy}{dt}$.

Rewrites the equation as:

$v + t\frac{dv}{dt} = v + \sqrt{(1+v^2)}$ where $v = \frac{y}{t}$ and reduces it to:

$$\frac{dv}{\sqrt{1-v^2}} = \frac{dt}{t}$$

Solves the differential equation as:

$$\sin^{-1}(v) = \ln|t| + C$$

where C is an arbitrary constant.

Substitues the value of v as $\frac{y}{t}$ and gets the solution as:

$$\sin^{-1}\left(\frac{y}{t}\right) = \ln|t| + C$$

Substitues the value of t as $\sin x$ and gets the final solution as:

$$\sin^{-1}\left(\frac{y}{\sin x}\right) = \ln|\sin x| + C$$

2. Takes $F = (x, y) = \frac{y}{x} + \cos\frac{y}{x}$ and finds $F(\lambda x, \lambda y)$ as:

$$F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} + \cos\frac{\lambda y}{\lambda x}$$

$$\Rightarrow \quad F(\lambda x, \lambda y) = \lambda^0 F(x, y)$$

Hence, concluses that the given differential equation be homogeneous.

Takes $y = vx$ and differentiates it to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Rewrites the given differential equation using the above step as:

$$\sec v \, dv = \frac{1}{x} dx$$

Integrates the above equation as:

$$\log(\sec v + \tan v) + \log C = \log x$$

where, C is an arbitrary constant.

Simplifies the above equation as:

$$C(\sec v + \tan v) = x$$

Replaces $y = vx$ in the above equation to find the general solution of the given differential equation as:

$$x = C\left[\sec\frac{y}{x} + \tan\frac{y}{x}\right]$$

3. Uses the variable separable form to rewrite the given equation as:

$$\frac{dy}{y} = k\,dt$$

Integrates the above equation on both sides to get:

$\ln y = kt + c$, where c is a constant.

Write the above equation in terms of e as:

$$y = e^{(kt + c)}$$

$$\Rightarrow \quad y = e^{kt} \times e^{c}$$

Uses the given conditions to write:

At $t = 0$, $y = e^c = e^{10}$, where e^{10} is the initial population.

Uses the given condition to write:

$$\text{At } t = 5,\ y = \frac{3}{4}e^{10} = e^{5k} \times e^{10}$$

$$\Rightarrow e^{5k} = \frac{3}{4}$$

Takes the natural logarithm on both sides to find the value of k as –0.058. The working may look as follows:

$$5k = \ln 3 - \ln 4$$

$$\Rightarrow \quad 5k = 1.09 - 1.38$$

$$\Rightarrow \quad 5k = -0.29$$

$$\Rightarrow \quad k = \frac{-0.29}{5} = -0.058$$

Level - 2 ADVANCED COMPETENCY FOCUSED QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQs) (1 Marks)

1. **Option (C) is correct.**

Explanation: Given:

$$\sin x \frac{dy}{dx} + y\cos x = \tan x$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = \frac{\cos x}{\sin x} = \cot x$ and $Q(x) = \frac{\tan x}{\sin x}$

The integrating factor (IF) is given by:

$$e^{\int P(x)dx} = e^{\int \cot x dx}$$

2. **Option (B) is correct.**

Explanation: We are given the general solution:

$$y = \sin x + \cos x + C$$

Differentiating both sides twice:

1st derivative:

$$\frac{dy}{dx} = \cos x - \sin x$$

3. **Option (D) is correct.**

Explanation: Given the differential equation:

$$\frac{d^2y}{dx^2} + y = 0$$

The general solution of this second-order homogeneous differential equation is:

$$y = A \sin x + B \cos x$$

Checking the given solutions:

Bulbul: $y = \sin x$ (By taking $A \in \mathbb{R}$ and $B = 0$)

Ipsita: $y = \cos x$ (Correct by taking $A = 0$ and $B \in \mathbb{R}$)

Sagarika: $y = \sin x + \cos x$ (This satisfies the general solution with $A, B \in R$, so it's correct.)

ASSERTION-REASON QUESTIONS (1 Marks)

1. **Option (A) is correct.**

Explanation: Assertion is true. The cooling of a hot body follows a first-order linear differential equation, commonly used to model heat transfer based on Newton's Law of Cooling.

Reason is also true. Newton's Law of Cooling is mathematically expressed as:

$$\frac{dT}{dt} = -k(T - T_a)$$

Where T = temperature of the body, T_a = ambient temperature, and k = positive constant.

Since the rate of temperature change is proportional to the difference in temperature between the object and its surroundings, this directly supports the assertion.

2. **Option (C) is correct.**

Explanation: Assertion is true. The differential equation $dy/dt = ky$ represents exponential growth when $k > 0$, since the rate of change of y is directly proportional to y itself.

Reason is false. In exponential growth, the rate of change is proportional to the quantity itself, not the square of the quantity.

3. **Option (A) is correct.**

Explanation: Assertion is true. Radioactive substances decay over time, and this behaviour can be mathematically described using a differential equation.

Reason is also true. The rate of decay of a radioactive substance is proportional to the current amount of the substance.

4. **Option (A) is correct.**

Explanation: Assertion is true.A first-order differential equation has the form:

$\frac{dy}{dx} = f(x, y)$. Its general solution contains one arbitrary constant, such as C, because it has only one integration step.

Reason is also true. In general, a differential equation of order n will have n arbitrary constants in its general solution. This is because each integration introduces one constant.

VERY SHORT ANSWER TYPE QUESTIONS (2 Marks)

1. Writes the derivative as :

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{x^2}$$

Rewrites both sides of the equation as:

$$\int dy = \int \frac{1}{(1+x^2)} dx - \int \frac{1}{x^2} dx$$

Integrates the above equation to get the function as:

$$y = \tan^{-1}(x) + \frac{1}{x} + C$$

2. Rewrites the given differential equation as $y dy = e^{3x} dx$.

Concludes that Sumit's general solution is incorrect by integrating both sides of the above equation to find the general solution as:

$y^2 = \frac{2}{3}e^{3x} + C$, where C is the arbitrary constant.

3. Writes that the statement is false.

Gives a reason. For example, the order of the given differential equation is 4, but its degree is 2.

4. Writes that the statement is false.

Gives a reason. The number of arbitrary constants in the general solution of a differential equation is determined by its order, not its degree. Since the order of the given equation is 4, it will have only 4 arbitrary constants.

5. Rewrites the given equation in terms of arbitrary constants as:

$$y = C_1 \operatorname{cosec}(x + C_2) - C_3 e^2 x$$

where, $C_1 = (R - S)$, $C_2 = T$ and $C_3 = Ue^{-V}$

Writes that since the order of a differential equation is same as the number of arbitrary constants, the order of the given differential equation is 3.

SHORT ANSWER TYPE QUESTIONS (3 Marks)

1. Rewrites the given differential equation as $\frac{dy}{dx} = 3 - 2\frac{y}{x}$.

Takes $y = vx$ and differentiates it to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Rewrites the differential equation using the substitution in the above step as:

$$\frac{dv}{(1-v)} = 3\frac{dx}{x}$$

Integrates the above equation as:

$-\log(1 - v) + \log |C_1| = 3 \log |x|$

where C_1 is the arbitrary constant.

(Award full marks for equivalent appropriate answers.)

Simplifies the equation further and substitutes v as $\frac{y}{x}$ to obtain the general solution of the given differential equation as:

$$\left|\frac{C_1}{1-v}\right| = x^3$$

$$\Rightarrow \quad \pm C_1 \cdot \frac{x}{(x-y)} = x^3$$

$$\Rightarrow x^3 = \frac{x}{(x-y)}C, \text{ where } C = \pm C_1$$

(Award full marks for equivalent appropriate answers.)

Concludes that:

$$f(x, y) = \frac{x}{(x-y)}$$

2. Solves the differential equation:

$$\int \frac{dN}{N} = -k\int dt$$

and writes $N = Ce^{-kt}$ as the solution where C is an arbitrary constant.

Writes that, at $t = 0$, $N = 45$ g and hence finds C as 45.

Calculates k as $\frac{0.693}{3} = 0.231$

Calculates the quantity of the radioactive material left after 10 days as:

$$N = 45e^{-0.231 \times 10} = 45 \times 0.099 = 4.45 \text{ g}$$

3. Rewrites the given differential equation as:

$$\int dP = -a\int (T - t)dt$$

Integrates the above equation as:

$$P(t) = \frac{at^2}{2} - atT + C$$

where C is an arbitrary constant.

Writes that, at $t = 0$, $P(t) = y$ so $C = y$.

Finds the value of the car as:

$$P(T) = \frac{aT^2}{2} - aT^2 + y = y - \frac{aT^2}{2}$$

CASE BASED QUESTIONS (4 Marks)

1. (i)
$$\frac{dP}{dt} = kP$$
$$\Rightarrow \int \frac{dP}{P} = \int k\, dt$$
$$\Rightarrow \log P = kt + C \text{ or } P = e^{kt + C}$$

(ii) $\log P = kt + C$

when $t = 0$, $P = 1000 \Rightarrow C = \log 1000$

when $t = 1$, $P = 2000 \Rightarrow \log 2000 = k + \log 1000$

$\Rightarrow k = \log 2$

2. (i) Models the situation and rearranges terms of form a linear differential equation as follows:
$$0.05\frac{dv}{dt} = -0.4v - 0.5$$
$$\Rightarrow \frac{dv}{dt} + 8v = -10$$

(ii) Considering the obtained equation as linear of the form $\frac{dy}{dx} + Py = Q$ with $P = 8$ and hence takes the integrating factor as:
$$e^{\int 8dt} = e^{8t}$$

Multiplies the differential equation by the integrating factor as follows:
$$e^{8t}\frac{dv}{dt} + 8ve^{8t} = -10e^{8t}$$
$$\Rightarrow \frac{d}{dt}(ve^{8t}) = -10e^{8t}$$

Integrates both sides to obtain the general solution of the differential equation as follows:
$$\int \frac{d}{dt}(ve^{8t}) = -10\int e^{8t}\, dt$$
$$\Rightarrow ve^{8t} = \frac{-10}{8}e^{8t} + C$$
$$\Rightarrow v = -1.25 + Ce^{-8t}$$

where C is the constant of integration.

Uses the initial condition $v(0) = 10$ m/s to find the value of C as follows:
$$10 = -1.25 + C$$
$$\Rightarrow C = 11.25$$

Hence, written the expression for the velocity of the ball as a function of time as follows:
$$v = -1.25 + 11.25e^{-8t}$$

3. (i) Identifies that the rod being heated is R_1 and finds the rate of change of temperature at any distance from one end of R_1 as:
$$\frac{dT}{dx} = \frac{d}{dx}(16 - x)x$$
$$= \frac{d}{dx}(16x - x^2)$$
$$= 16 - 2x$$

Finds the mid-point of the rod as $x = 8$ m.

Finds the rate of change of temperature at the mid point of R_1 as:
$$\left.\frac{dT}{dx}\right|_{\text{at } x = 8} = 16 - 2(8) = 0$$

(ii) Identifies that the rod being cooled is R_2 and finds the rate of change of temperature at any distance x m as:
$$\frac{dT}{dx} = \frac{d}{dx}(x - 12)x$$
$$= \frac{d}{dx}(x^2 - 12x)$$
$$= 2x - 12$$

Equates $\frac{dT}{dx}$ to 0 to get the critical point as $x = 6$.

Finds the second derivative of T as:
$$\frac{d^2T}{dx^2} = 2$$

And concludes that at $x = 6$ m, the rod has minimum temperature as $\left.\frac{d^2T}{dx^2}\right|_{\text{at } x = 0}$

Finds the minimum temperature attained by the rod R_2 as
$$T(6) = (6 - 12)6 = -36\ °C$$

4. (i) $(x^2 - y^2)dx + 2xydy = 0$
$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$
$$= \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)}$$
$$= g\left(\frac{y}{x}\right)$$

(ii)
$$y = vx$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$
$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$
$$x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$
$$\Rightarrow \int \frac{2v}{1 + v^2}dv = -\int \frac{dx}{x}$$
$$\Rightarrow \log|1 + v^2| + \log|x| = \log C$$
$$\text{or} \quad x\left(1 + \frac{y^2}{x^2}\right) = C$$
$$\text{or} \quad x^2 + y^2 = Cx$$

5. **(i)** Uses the information given to find the rate of change of quantity of salt as follows:

$$\frac{dQ_1}{dt} = 0 - 3000 \times \frac{Q_1}{20000 + 5000t - 3000t}$$

$$= \frac{-3Q_1}{20 + 2t}$$

(ii) Separates the variables of the differential equation as follows:

$$\frac{dQ_1}{dt} = \frac{-3Q_1}{20 + 2t}$$

$$\Rightarrow \quad \frac{dQ_1}{-3Q_1} = \frac{dt}{20 + 2t}$$

Integrates both sides to obtain the following:

$$\int \frac{dQ_1}{-3Q_1} = \int \frac{dt}{20 + 2t}$$

$$\Rightarrow \quad -\frac{1}{3}\ln Q_1 = \frac{1}{2}\ln(10 + t) + C$$

where C is an arbitrary constant.

(iii) Takes initial condition $Q_t(0) = 2000$ to obtain C as follows:

$$-\frac{1}{3}\ln 2000 = \frac{1}{2}\ln 10 + C$$

$$C = -\frac{1}{3}\ln 2 - \frac{3}{2}\ln 10$$

Frames the equation as:

$$-\frac{1}{3}\ln Q_t = \frac{1}{2}\ln(10 + t) - \frac{1}{3}\ln 2 - \frac{3}{2}\ln 10$$

LONG ANSWER TYPE QUESTIONS (5 Marks)

1.
$$\frac{dV}{V} = k\,dt$$

Integrate both sides:

$$\log|V| = kt + C$$

$\Rightarrow$ $V(t) = Ae^{kt}$ (where $A = e^C$)

Use $V(0) = 200$

$$200 = Ae^{k.0} = A \Rightarrow A = 200$$

So, $V(t) = 200e^{kt}$

Use $V(1) = 300$

$$300 = 200e^k \Rightarrow e^k = \frac{3}{2} \Rightarrow k = \log\left(\frac{3}{2}\right)$$

$$\therefore \quad V(2) = 200e^{2k} = 200.(e^k)^2 = 200.\left(\frac{3}{2}\right)^2$$

$$= 200.\frac{9}{4} = 450 \text{ liters}$$

2.
$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = kdt$$

Integrating both sides:

$$\log|P| = kt + C \Rightarrow P(t) = Ae^{kt}$$

Use initial condition $P(0) = 20000$ to find A:

$$P(0) = Ae^{k.0} = A \Rightarrow A = 20000$$

So, $P(t) = 20000.e^{kt}$

Use $P(2) = 25000$ to find k

$$25000 = 20000 \,.\, e^{2k}$$

$$\Rightarrow \quad \frac{5}{4} = e^{2k} \Rightarrow 2k = \log\left(\frac{5}{4}\right)$$

$$\Rightarrow \quad k = \frac{1}{2}\log\left(\frac{5}{4}\right)$$

P(5) (i.e. in 2025)

$$P(5) = 20000 \,.\, e^{5k} = 20000 \,.\left(e^{2k}\right)^{\frac{5}{2}}$$

$$= 20000 \,.\left(\frac{5}{4}\right)^{\frac{5}{2}}$$

$$\left(\frac{5}{4}\right)^{\frac{5}{2}} = (1.25)^{2.5}$$

$$\approx 1.25^2 \times \sqrt{1.25}$$

$$\approx 1.5625 \,.\, 1.1180 \approx 1.747$$

$$P(5) = 20000 \times 1.747 = 34{,}940$$